Notes on generating functions

1 Basics

The short version: A generating function represents objects of weight n with z^n, and adds all the objects you have up to get a sum $a_0z^0+a_1z^1+a_2z^2+\dots$, where each a_n counts the number of different objects of weight n. If you are very lucky (or constructed your set of objects by combining simpler sets of objects in certain straightforward ways) there will be some compact expression that is expands to this horrible sum but is easier to right down. Such compact expressions are called **generating functions**, and manipulating them algebraically gives an alternative to actually knowing HowToCount.

1.1 A simple example

We are given some initial prefixes for words: `qu`, `s`, and `t`; some vowels to put in the middle: `a`, `i`, and `oi`; and some suffixes: `d`, `ff`, and `ck`, and we want to calculate the number of words we can build of each length.

One way is to generate all 27 words[1] and sort them by length:

```
sad sid tad tid
quad quid sack saff sick siff soid tack taff tick tiff toid
quack quaff quick quiff quoid soick soiff toick toiff
quoick quoiff
```

This gives us 4 length-3 words, 12 length-4 words, 9 length-5 words, and 2 length-6 words. This is probably best done using a computer, and becomes expensive if we start looking at much larger lists.

An alternative is to solve the problem by judicious use of algebra. Pretend that each of our letters is actually a variable, and that when we concatenate `qu`, `oi`, and `ck` to make `quoick`, we are really multiplying the variables using our usual notation. Then we can express all 27 words as the product (`qu`+`s`+`t`)(`a`+`i`+`oi`)(`d`+`ff`+`ck`). But we don't care about the exact set of words, we just want to know how many we get of each length.

[1] We are using *word* in the combinatorial sense of a finite sequence of letters (possibly even the empty sequence) and not the usual sense of a finite, nonempty sequence of letters that actually make sense.

So now we do the magic trick: we replace every variable we've got with a single variable z. For example, this turns `quoick` into $zzzzzz = z^6$, so we can still find the length of a word by reading off the exponent on z. But we can also do this before we multiply everything out, getting

$$\begin{aligned}(zz + z + z)(z + z + zz)(z + zz + zz) &= (2z + z^2)(2z + z^2)(z + 2z^2)\\ &= z^3(2 + z)^2(1 + 2z)\\ &= z^3(4 + 4z + z^2)(1 + 2z)\\ &= z^3(4 + 12z + 9z^2 + 2z^3)\\ &= 4z^3 + 12z^4 + 9z^5 + 2z^6.\end{aligned}$$

We can now read off the number of words of each length directly off the coefficients of this polynomial.

1.2 Why this works

In general, what we do is replace any object of **weight** 1 with z. If we have an object with weight n, we think of it as n weight-1 objects stuck together, i.e., z^n. Disjoint unions are done using addition as in simple counting: $z + z^2$ represents the choice between a weight-1 object and a weight-2 object (which might have been built out of 2 weight-1 objects), while $12z^4$ represents a choice between 12 different weight-4 objects. The trick is that when we multiply two expressions like this, whenever two values z^k and z^l collide, the exponents add to give a new value z^{k+l} representing a new object with total weight $k+l$, and if we have something more complex like $(nz^k)(mz^l)$, then the coefficients multiply to give $(nm)z^{k+l}$ different weight $(k + l)$ objects.

For example, suppose we want to count the number of robots we can build given 5 choices of heads, each of weight 2, and 6 choices of bodies, each of weight 5. We represent the heads by $5z^2$ and the bodies by $6z^5$. When we multiply these expressions together, the coefficients multiply (which we want, by the product rule) and the exponents add: we get $5z^2 \cdot 6z^5 = 30z^7$ or 30 robots of weight 7 each.

The real power comes in when we consider objects of different weights. If we add to our 5 weight-2 robot heads two extra-fancy heads of weight 3, and compensate on the body side with three new lightweight weight-4 bodies, our new expression is $(5z^2 + 2z^3)(3z^4 + 6z^5) = 15z^6 + 36z^7 + 12z^8$, giving a possible 15 weight-6 robots, 36 weight-7 robots, and 12 weight-8 robots. The rules for multiplying polynomials automatically tally up all the different cases for us.

This trick even works for infinitely-long polynomials that represent infinite series (such "polynomials" are called formal power series). Even though there might be infinitely many ways to pick three natural numbers, there are only finitely many ways to pick three natural numbers whose sum is 37. By computing an appropriate formal power series and extracting the coefficient from the z^{37} term, we can figure out exactly how many ways there are. This works best, of course, when we don't have to haul around an entire infinite series, but can instead represent it by some more compact function whose expansion gives the desired series. Such a function is called a **generating function**, and manipulating generating functions can be a powerful alternative to creativity in making combinatorial arguments.

1.3 Formal definition

Given a sequence $a_0, a_1, a_2, \ldots$, its **generating function** $F(z)$ is given by the sum

$$F(z) = \sum_{i=0}^{\infty} a_i z^i.$$

A sum in this form is called a **formal power series**. It is "formal" in the sense that we don't necessarily plan to actually compute the sum, and are instead using the string of z^i terms as a long rack to store coefficients on.

In some cases, the sum has a more compact representation. For example, we have

$$\frac{1}{1-z} = \sum_{i=0}^{\infty} z^i,$$

so $1/(1-z)$ is the generating function for the sequence $1, 1, 1, \ldots$. This may let us manipulate this sequence conveniently by manipulating the generating function.

Here's a simple case. If $F(z)$ generates some sequence a_i, what does sequence b_i does $F(2z)$ generate? The i-th term in the expansion of $F(2z)$ will be $a_i(2z)^i = a_i 2^i z^i$, so we have $b_i = 2^i a_i$. This means that the sequence $1, 2, 4, 8, 16, \ldots$ has generating function $1/(1-2z)$. In general, if $F(z)$ represents a_i, then $F(cz)$ represents $c^i a_i$.

What else can we do to F? One useful operation is to take its derivative with respect to z. We then have

$$\frac{d}{dz} F(z) = \sum_{i=0}^{\infty} a_i \frac{d}{dz} z^i = \sum_{i=0}^{\infty} a_i i z^{i-1}.$$

This *almost* gets us the representation for the series $i a_i$, but the exponents on the z's are off by one. But that's easily fixed:

$$z \frac{d}{dz} F(z) = z \sum_{i=0}^{\infty} a_i i z^{i-1} = \sum_{i=0}^{\infty} a_i i z^i.$$

So the sequence $0, 1, 2, 3, 4, \ldots$ has generating function

$$z \frac{d}{dz} \frac{1}{1-z} = \frac{z}{(1-z)^2},$$

and the sequence of squares $0, 1, 4, 9, 16, \ldots$ has generating function

$$z \frac{d}{dz} \frac{z}{(1-z)^2} = \frac{z}{(1-z)^2} + \frac{2z^2}{(1-z)^3}.$$

As you can see, some generating functions are prettier than others.

(We can also use integration to divide each term by i, but the details are messier.)

Another way to get the sequence $0, 1, 2, 3, 4, \ldots$ is to observe that it satisfies the recurrence:

- $a_0 = 0$.
- $a_{n+1} = a_n + 1 (\forall n \in \mathbb{N})$.

A standard trick in this case is to multiply each of the $\forall i$ bits by z^n, sum over all n, and see what happens. This gives $\sum a_{n+1} z^n = \sum a_n z^n + \sum z^n = \sum a_n z^n + 1/(1-z)$. The first term on the right-hand side is the generating function for a_n, which we can call $F(z)$ so we don't have to keep writing it out. The second term is just the generating function for $1, 1, 1, 1, 1, \ldots$. But what about the left-hand side? This is almost the same as $F(z)$, except the coefficients don't match up with the exponents. We can fix this by dividing $F(z)$ by z, after carefully subtracting off the a_0 term:

$$\begin{aligned}
(F(z) - a_0)/z &= \left(\sum_{n=0}^{\infty} a_n z^n - a_0 \right) / z \\
&= \left(\sum_{n=1}^{\infty} a_n z^n \right) / z \\
&= \sum_{n=1}^{\infty} a_n z^{n-1} \\
&= \sum_{n=0}^{\infty} a_{n+1} z^n.
\end{aligned}$$

So this gives the equation $(F(z) - a_0)/z = F(z) + 1/(1-z)$. Since $a_0 = 0$, we can rewrite this as $F(z)/z = F(z) + 1/(1-z)$. A little bit of algebra turns this into $F(z) - zF(z) = z/(1-z) or F(z) = z/(1-z)^2$.

Yet another way to get this sequence is construct a collection of objects with a simple structure such that there are exactly n objects with weight n. One way to do this is to consider strings of the form $a^+ b^*$ where we have at least one a followed by zero or more b's. This gives n strings of length n, because we get one string for each of the 1 through n a's we can put in (an example would be abb, aab, and aaa for $n = 3$). We can compute the generating function for this set because to generate each string we must pick in order:

- One initial a. Generating function $= z$.
- Zero or more a's. Generating function $= 1/(1-z)$.
- Zero or more b's. Generating function $= 1/(1-z)$.

Taking the product of these gives $z/(1-z)^2$, as before.

This trick is useful in general; if you are given a generating function $F(z)$ for a_n, but want a generating function for $b_n = \sum_{k \le n} a_k$, allow yourself to pad each weight-k object out to weight n in exactly one way using $n-k$ junk objects, i.e. multiply $F(z)$ by $1/(1-z)$.

2 Some standard generating functions

Here is a table of some of the most useful generating functions.

$$
\begin{aligned}
\frac{1}{1-z} &= \sum_{i=0}^{\infty} z^i \\
\frac{z}{(1-z)^2} &= \sum_{i=0}^{\infty} i z^i \\
(1+z)^n &= \sum_{i=0}^{\infty} \binom{n}{i} z^i = \sum_{i=0}^{n} \binom{n}{i} z^i \\
\frac{1}{(1-z)^n} &= \sum_{i=0}^{\infty} \binom{n+i-1}{i} z^i
\end{aligned}
$$

Of these, the first is the most useful to remember (it's also handy for remembering how to sum geometric series). All of these equations can be proven using the binomial theorem.

3 More operations on formal power series and generating functions

Let $F(z) = \sum_i a_i z^i$ and $G(z) = \sum_i b_i z^i$. Then their sum $F(z) + G(z) = \sum_i (a_i + b_i) z^i$ is the generating function for the sequence $(a_i + b_i)$. What is their product $F(z)G(z)$?

To compute the i-th term of $F(z)G(z)$, we have to sum over all pairs of terms, one from F and one from G, that produce a z^i factor. Such pairs of terms are precisely those that have exponents that sum to i. So we have

$$F(z)G(z) = \sum_{i=0}^{\infty} \left(\sum_{j=0}^{i} a_j b_{j-i} \right) z^i.$$

As we've seen, this equation has a natural combinatorial interpretation. If we interpret the coefficient a_i on the i-th term of $F(z)$ as counting the number of "a-things" of weight i, and the coefficient b_i as the number of "b-things" of weight i, then the i-th coefficient of $F(z)G(z)$ counts the number of ways to make a combined thing of total weight i by gluing together an a-thing and a b-thing.

As a special case, if $F(z) = G(z)$, then the i-th coefficient of $F(z)G(z) = F^2(z)$ counts how many ways to make a thing of total weight i using two "a-things", and $F^n(z)$ counts how many ways (for each i) to make a thing of total weight i using n "a-things". This gives us an easy combinatorial proof of a special case of the binomial theorem:

$$(1+x)^n = \sum_{i=0}^{\infty} \binom{n}{i} x^i.$$

Think of the left-hand side as the generating function $F(x) = 1+x$ raised to the n-th power. The function F by itself says that you have a choice between one weight-0 object or one weight-1 object. On the right-hand side the i-th coefficient counts how many ways you can put together a total of i weight-1 objects given n to choose from—so it's $\binom{n}{i}$.

4 Counting with generating functions

The product formula above suggests that generating functions can be used to count combinatorial objects that are built up out of other objects, where our goal is to count the number of objects of each possible non-negative integer "weight" (we put "weight" in scare quotes because we can make the "weight" be any property of the object we like, as long as it's a non-negative integer—a typical choice might be the size of a set, as in the binomial theorem example above). There are five basic operations involved in this process; we've seen two of them already, but will restate them here with the others.

Throughout this section, we assume that $F(z)$ is the generating function counting objects in some set A and $G(z)$ the generating function counting objects in some set B.

4.1 Disjoint union

Suppose $C = A \cup B$ and A and B are disjoint. Then the generating function for objects in C is $F(z) + G(z)$.

Example: Suppose that A is the set of all strings of zero or more letters x, where the weight of a string is just its length. Then $F(z) = 1/(1-z)$, since there is exactly one string of each length and the coefficient a_i on each z^i is always 1. Suppose that B is the set of all strings of zero or more letters y and/or z, so that $G(z) = 1/(1-2z)$ (since there are now 2^i choices of length-i strings). The set C of strings that are either (a) all x's or (b) made up of y's, z's, or both, has generating function $F(z) + G(z) = 1/(1-z) + 1/(1-2z)$.

4.2 Cartesian product

Now let $C = A \times B$, and let the weight of a pair $(a, b) \in C$ be the sum of the weights of a and b. Then the generating function for objects in C is $F(z)G(z)$.

Example: Let A be all-x strings and B be all-y or all-z strings, as in the previous example. Let C be the set of all strings that consist of zero or more x's followed by zero or more y's and/or z's. Then the generating function for C is $F(z)G(z) = \frac{1}{(1-z)(1-2z)}$.

4.3 Repetition

Now let C consists of all finite sequences of objects in A, with the weight of each sequence equal to the sum of the weights of its elements (0 for an empty sequence). Let $H(z)$ be the generating function for C. From the preceding rules we have

$$H = 1 + F + F^2 + F^3 + \dots = \frac{1}{1-F}.$$

This works best when $H(0) = 0$; otherwise we get infinitely many weight-0 sequences. It's also worth noting that this is just a special case of substitution (see below), where our "outer" generating function is $1/(1-z)$.

4.3.1 Example: $(0|11)*$

Let $A = \{0, 11\}$, and let C be the set of all sequences of zeros and ones where ones occur only in even-length runs. Then the generating function for A is $z + z^2$ and the generating function for C is $1/(1-z-z^2)$. We can extract exact coefficients from this generating function using the techniques below.

4.3.2 Example: sequences of positive integers

Suppose we want to know how many different ways there are to generate a particular integer as a sum of positive integers. For example, we can express 4 as 4, $3+1$, $2+2$, $2+1+1$, $1+1+1+1$, $1+1+2$, $1+2+1$, or $1+3$, giving 8 different ways.

We can solve this problem using the repetition rule. Let $F = z/(1-z)$ generate all the positive integers. Then

$$\begin{aligned}
H &= \frac{1}{1-F} \\
&= \frac{1}{1-\frac{z}{1-z}} \\
&= \frac{1-z}{(1-z)-z} \\
&= \frac{1-z}{1-2z}.
\end{aligned}$$

We can get exact coefficients by observing that

$$\begin{aligned}
\frac{1-z}{1-2z} &= \frac{1}{1-2z} - \frac{z}{1-2z} \\
&= \sum_{n=0}^{\infty} 2^n z^n - \sum_{n=0}^{\infty} 2^n z^{n+1} \\
&= \sum_{n=0}^{\infty} 2^n z^n - \sum_{n=1}^{\infty} 2^{n-1} z^n \\
&= 1 + \sum_{n=1}^{\infty} (2^n - 2^{n-1}) z^n \\
&= 1 + \sum_{n=1}^{\infty} 2^{n-1} z^n.
\end{aligned}$$

This means that there is 1 way to express 0 (the empty sum), and 2^{n-1} ways to express any larger value n (e.g. $2^{4-1} = 8$ ways to express 4).

Once we know what the right answer is, it's not terribly hard to come up with a combinatorial explanation. The quantity 2^{n-1} counts the number of subsets of an $(n-1)$-element set. So imagine that we have $n-1$ places and we mark some subset of them, plus add an extra mark at the end; this might give us a pattern like `XX-X`. Now for each sequence of places ending with a mark we replace it with the number of places (e.g. `XX-X` $= 1, 1, 2$, `X--X-X---X` $= 1, 3, 2, 4$). Then the sum of the numbers we get is equal to n, because it's just counting the total length of the sequence by dividing it up at the marks and the adding the pieces back together. The value 0 doesn't fit this pattern (we can't put in the extra mark without getting a sequence of length 1), so we have 0 as a special case again.

If we are very clever, we might come up with this combinatorial explanation from the beginning. But the generating function approach saves us from having to be clever.

4.4 Pointing

This operation is a little tricky to describe. Suppose that we can think of each weight-k object in A as consisting of k items, and that we want to count not only how many weight-k objects there are, but how many ways we can produce a weight-k object where one of its k items has a special mark on it. Since there are k different items to choose for each weight-k object, we are effectively multiplying the count of weight-k objects by k. In generating function terms, we have

$$H(z) = z\frac{d}{dz}F(z).$$

Repeating this operation allows us to mark more items (with some items possibly getting more than one mark). If we want to mark n distinct items in each object (with distinguishable marks), we can compute

$$H(z) = z^n \frac{d^n}{dz^n} F(z),$$

where the repeated derivative turns each term $a_i z^i$ into $a_i i(i-1)(i-2)\dots(i-n+1)z^{i-n}$ and the z^n factor fixes up the exponents. To make the marks indistinguishable (i.e., we don't care what order the values are marked in), divide by $n!$ to turn the extra factor into $\binom{i}{n}$.

(If you are not sure how to take a derivative, look at HowToDifferentiate.)

Example: Count the number of finite sequences of zeros and ones where exactly two digits are underlined. The generating function for {0,1} is $2z$, so the generating function for sequences of zeros and ones is $F = 1/(1-2z)$ by the repetition rule. To mark two digits with indistinguishable marks, we need to compute

$$\frac{1}{2} z^2 \frac{d^2}{dz^2} \frac{1}{1-2z} = \frac{1}{2} z^2 \frac{d}{dz} \frac{2}{(1-2z)^2} = \frac{1}{2} z^2 \frac{8}{(1-2z)^3} = \frac{4z^2}{(1-2z)^3}.$$

4.5 Substitution

Suppose that the way to make a C-thing is to take a weight-k A-thing and attach to each its k items a B-thing, where the weight of the new C-thing is the sum of the weights of the B-things. Then the generating function for C is the composition $F(G(z))$.

Why this works: Suppose we just want to compute the number of C-things of each weight that are made from some single specific weight-k A-thing. Then the generating function for this quantity is just $(G(z))^k$. If we expand our horizons to include all a_k weight-k A-things, we have to multiply by a_k to get $a_k(G(z))^k$. If we further expand our horizons to include A-things of all different weights, we have to sum over all k:

$$\sum_{k=0}^{\infty} a_k (G(z))^k.$$

But this is just what we get if we start with $F(z)$ and substitute $G(z)$ for each occurrence of z, i.e. if we compute $F(G(z))$.

4.5.1 Example: bit-strings with primes

Suppose we let A be all sequences of zeros and ones, with generating function $F(z) = 1/(1-2z)$. Now suppose we can attach a single or double prime to each 0 or 1, giving $0'$ or $0''$ or $1'$ or $1''$, and we want a generating function for the number of distinct primed bit-strings with n attached primes. The set $\{', ''\}$ has generating function $G(z) = z + z^2$, so the composite set has generating function $F(z) = 1/(1-2(z+z^2)) = 1/(1-2z-2z^2)$.

4.5.2 Example: (0—11)* again

The previous example is a bit contrived. Here's one that's a little more practical, although it involves a brief digression into **multivariate generating functions**. A multivariate generating function $F(x, y)$ generates a series $\sum_{ij} a_{ij} x^i y^j$, where a_{ij} counts the number of things that have i x's and j y's. (There is also the obvious generalization to more than two variables). Consider the multivariate generating function for the set {0,1}, where x counts zeros and y counts ones: this is just $x + y$. The multivariate generating function for sequences of zeros and ones is $1/(1-x-y)$ by the repetition rule. Now suppose that each 0 is left intact but each 1 is replaced by 11, and we want to count the total number of strings by length, using z as our series variable. So we substitute z for x and z^2 for y (since each y turns into a string of length 2), giving $1/(1 - z - z^2)$. This gives another way to get the generating function for strings built by repeating 0 and 11.

5 Generating functions and recurrences

What makes generating functions particularly useful for algorithm analysis is that they directly solve recurrences of the form $T(n) = aT(n - 1) + bT(n - 2) + f(n)$ (or similar recurrences with more T terms on the right-hand side), provided we have a generating function $F(z)$ for $f(n)$. The idea is that there exists some generating function $G(z)$ that describes the entire sequence of values $T(0), T(1), T(2), \ldots$, and we just need to solve for it by restating the recurrence as an equation about G. The left-hand side will just turn into G. For the right-hand side, we need to shift $T(n-1)$ and $T(n-2)$ to line up right, so that the right-hand side will correctly represent the sequence $T(0), T(1), aT(0) + aT(1) + F(2)$, etc. It's not hard to see that the generating function for the sequence $0, T(0), T(1), T(2), \ldots$ (corresponding to the $T(n - 1)$ term) is just $zG(z)$, and similarly the sequence $0, 0, T(1), T(2), T(3), \ldots$ (corresponding to the $T(n - 2)$ term) is $z^2G(z)$. So we have (being very careful to subtract out extraneous terms at for $i = 0$ and $i = 1$):

$$G = az(G - T(0)) + bz^2G + (F - f(0) - zf(1)) + T(0) + zT(1),$$

and after expanding F we can in principle solve this for G as a function of z.

5.1 Example: A Fibonacci-like recurrence

Let's take a concrete example. The Fibonacci-like recurrence

$$T(n) = T(n - 1) + T(n - 2), T(0) = 1, T(1) = 1,$$

becomes

$$G = (zG - z) + z^2G + 1 + z.$$

(here $F = 0$).

Solving for G gives

$$G = 1/(1 - z - z^2).$$

Unfortunately this is not something we recognize from our table, although it has shown up in a couple of examples. (Exercise: *Why* does the recurrence $T(n) = T(n-1) + T(n-2)$ count the number of strings built from 0 and 11 of length n?) In the next section we show how to recover a closed-form expression for the coefficients of the resulting series.

6 Recovering coefficients from generating functions

There are basically three ways to recover coefficients from generating functions:

1. Recognize the generating function from a table of known generating functions, or as a simple combination of such known generating functions. This doesn't work very often but it is possible to get lucky.

2. To find the k-th coefficient of $F(z)$, compute the k-th derivative $d^k/dz^k F(z)$ and divide by $k!$ to shift a_k to the z^0 term. Then substitute 0 for z. For example, if $F(z) = 1/(1-z)$ then $a_0 = 1$ (no differentiating), $a_1 = 1/(1-0)^2 = 1$, $a_2 = 1/(1-0)^3 = 1$, etc. This usually only works if the derivatives have a particularly nice form or if you only care about the first couple of coefficients (it's particularly effective if you only want a_0).

3. If the generating function is of the form $1/Q(z)$, where Q is a polynomial with $Q(0) \neq 0$, then it is generally possible to expand the generating function out as a sum of terms of the form $P_c/(1 - z/c)$ where c is a root of Q (i.e. a value such that $Q(c) = 0$). Each denominator P_c will be a constant if c is not a repeated root; if c is a repeated root, then P_c can be a polynomial of degree up to one less than the multiplicity of c. We like these expanded solutions because we recognize $1/(1 - z/c) = \sum_i c^{-i} z^i$, and so we can read off the coefficients a_i generated by $1/Q(z)$ as an appropriately weighted some of c_1^{-i}, c_2^{-i}, etc., where the c_j range over the roots of Q.

Example: Take the generating function $G = 1/(1 - z - z^2)$. We can simplify it by factoring the denominator: $1 - z - z^2 = (1 - az)(1 - bz)$ where $1/a$ and $1/b$ are the solutions to the equation $1 - z - z^2 = 0$; in this case $a = (1 + \sqrt{5})/2$, which is approximately 1.618 and $b = (1 - \sqrt{5})/2$, which is approximately -0.618. It happens to be the case that we can always expand $1/P(z)$ as $A/(1 - az) + B(1 - bz)$ for some constants A and B whenever P is a degree 2 polynomial with constant coefficient 1 and distinct roots a and b, so

$$G = A/(1 - az) + B/(1 - bz),$$

and here we can recognize the right-hand side as the sum of the generating functions for the sequences $A \cdot a^i$ and $B \cdot b^i$. The $A \cdot a^i$ term dominates, so we have that $T(n) = \Theta(a^n)$, where a is approximately 1.618. We can also solve for A and B exactly to find an exact solution if desired.

A rule of thumb that applies to recurrences of the form $T(n) = a_1 T(n-1) + a_2 T(n-2) + \dots a_k T(n-k) + f(n)$ is that unless f is particularly large, the solution is usually exponential in $1/x$, where x is the smallest root of the polynomial $1 - a_1 z$-$a_2 z^2 \dots$-$a_k z^k$. This can be used to get very quick estimates of the solutions to such recurrences (which can then be proved without fooling around with generating functions).

Exercise: What is the exact solution if $T(n) = T(n-1) + T(n-2) + 1$? Or if $T(n) = T(n-1) + T(n-2) + n$?

6.1 Partial fraction expansion and Heaviside's cover-up method

There is a nice trick for finding the numerators in a partial fraction expansion. Suppose we have

$$\frac{1}{(1-az)(1-bz)} = \frac{A}{1-az} + \frac{B}{1-bz}.$$

Multiply both sides by $1 - az$ to get

$$\frac{1}{1-bz} = A + \frac{B(1-az)}{1-bz}.$$

Now plug in $z = 1/a$ to get

$$\frac{1}{1-b/a} = A + 0.$$

We can immediately read off A. Similarly, multiplying by $1 - bz$ and then setting $1 - bz$ to zero gets B. The method is known as the "cover-up method" because multiplication by $1 - az$ can be simulated by covering up $1 - az$ in the denominator of the left-hand side and all the terms that don't have $1 - az$ in the denominator in the right hand side.

The cover-up method will work in general whenever there are no repeated roots, even if there are many of them; the idea is that setting $1 - qz$ to zero knocks out all the terms on the right-hand side but one. With repeated roots we have to worry about getting numerators that aren't just a constant, so things get more complicated. We'll come back to this case below.

6.1.1 Example: A simple recurrence

Suppose $f(0) = 0, f(1) = 1$, and for $n \geq 2$, $f(n) = f(n-1) + 2f(n-2)$. Multiplying these equations by z^n and summing over all n gives a generating function

$$F(z) = \sum_{n=0} \infty f(n) z^n = 0 \cdot z^0 + 1 \cdot z^1 + \sum_{n=2}^{\infty} f(n-1) z^n + \sum_{n=2}^{\infty} 2f(n-2) z^n.$$

With a bit of tweaking, we can get rid of the sums on the RHS by converting them into copies of F:

$$\begin{aligned}
F(z) &= z + \sum_{n=2}^{\infty} f(n-1)z^n + 2\sum_{n=2}^{\infty} f(n-2)z^n \\
&= z + \sum_{n=1}^{\infty} f(n)z^{n+1} + 2\sum_{n=0}^{\infty} f(n)z^{n+2} \\
&= z + z\sum_{n=1}^{\infty} f(n)z^n + 2z^2\sum_{n=0}^{\infty} f(n)z^n \\
&= z + z(F(z) - f(0)z^0) + 2z^2F(z) \\
&= z + zF(z) + 2z^2F(z).
\end{aligned}$$

Now solve for $F(z)$ to get $F(x) = \frac{\{}{z}\}\{1 - z - 2z^2\} = \frac{\{}{z}\}\{(1+z)(1-2z)\} = z\left(\frac{\{}{A}\}\{1+z\} + \frac{\{}{B}\}\{1-2z\}\right)$, where we need to solve for A and B.

We can do this directly, or we can use the cover-up method. The cover-up method is easier. Setting $z = -1$ and covering up $1+z$ gives $A = 1/(1-2(-1)) = 1/3$. Setting $z = 1/2$ and covering up $1-2z$ gives $B = 1/(1+z) = 1/(1+1/2) = 2/3$. So we have

$$\begin{aligned}
F(z) &= \frac{(1/3)z}{1+z} + \frac{(2/3)z}{1-2z} \\
&= \sum_{n=0}^{\infty} \frac{(-1)^n}{3} z^{n+1} + \sum_{n=0}^{\infty} \frac{2 \cdot 2^n}{3} z^{n+1} \\
&= \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{3} z^n + \sum_{n=1}^{\infty} \frac{2^n}{3} z^n \\
&= \sum_{n=1}^{\infty} \left(\frac{2^n - (-1)^n}{3}\right) z^n.
\end{aligned}$$

This gives $f(0) = 0$ and, for $n \geq 1$, $f(n) = \frac{\{}{2}^{n-(-1)}n\}\{3\}$. It's not hard to check that this gives the same answer as the recurrence.

6.1.2 Example: Coughing cows

Let's count the number of strings of each length of the form `(M)*(O|U)*(G|H|K)*` where `(x|y)` means we can use x or y and * means we can repeat the previous parenthesized expression 0 or more times (see WikiPedia's article on regular expressions).

We start with a sequence of 0 or more M's. The generating function for this part is our old friend $1/(1-z)$. For the second part, we have two choices for each letter, giving $1/(1-2z)$. For the third part, we have $1/(1-3z)$. Since each part can be chosen independently of the other two, the generating function for all three parts together is just the product:

$$\frac{1}{(1-z)(1-2z)(1-3z)}.$$

Let's use the cover-up method to convert this to a sum of partial fractions. We have

$$\begin{aligned}
\frac{1}{(1-z)(1-2z)(1-3z)} &= \frac{\left(\frac{1}{(1-2)(1-3)}\right)}{1-z} + \frac{\left(\frac{1}{(1-\frac{1}{2})(1-\frac{3}{2})}\right)}{1-2z} + \frac{\left(\frac{1}{(1-\frac{1}{3})(1-\frac{2}{3})}\right)}{1-3z} \\
&= \frac{\frac{1}{2}}{1-z} + \frac{-4}{1-2z} + \frac{\frac{9}{2}}{1-3z}.
\end{aligned}$$

So the exact number of length-n sequences is $(1/2) - 4 \cdot 2^n + (9/2) \cdot 3^n$. We can check this for small n:

n	Formula	Strings
0	$1/2 - 4 + 9/2 = 1$	()
1	$1/2 - 8 + 27/2 = 6$	M, O, U, G, H, K
2	$1/2 - 16 + 81/2 = 25$	$MM, MO, MU, MG, MH, MK, OO, OU, OG, OH, OK, UO, UU, UG, UH, UK, G$
3	$1/2 - 32 + 243/2 = 90$	(exercise)$\ddot{\smile}$

6.1.3 Example: A messy recurrence

Let's try to solve the recurrence $T(n) = 4T(n-1) + 12T(n-2) + 1$ with $T(0) = 0$ and $T(1) = 1$.

Let $F = \sum T(n) z^n$.

Summing over all n gives

$$\begin{aligned}
F = \sum_{n=0}^{\infty} T(n) z^n &= T(0)z^0 + T(1)z^1 + 4\sum_{n=2}^{\infty} T(n-1)z^n + 12\sum_{n=2}^{\infty} T(n-2)z^n + \sum_{n=2}^{\infty} 1 \cdot z^n \\
&= z + 4z\sum_{n=1}^{\infty} T(n)z^n + 12z^2\sum_{n=0}^{\infty} T(n)z^n + z^2\sum_{n=0}^{\infty} z^n \\
&= z + 4z(F - T(0)) + 12z^2 F + \frac{z^2}{1-z} \\
&= z + 4zF + 12z^2 F + \frac{z^2}{1-z}.
\end{aligned}$$

Solving for F then gives

$$F = \frac{\left(z + \frac{z^2}{1-z}\right)}{1 - 4z - 12z^2}.$$

We want to solve this using partial fractions, so we need to factor $(1 - 4z - 12z^2) = (1 + 2z)(1 - 6z)$. This gives

$$
\begin{aligned}
F &= \frac{\left(z + \frac{z^2}{1-z}\right)}{(1+2z)(1-6z)} \\
&= \frac{z}{(1+2z)(1-6z)} + \frac{z^2}{(1-z)(1+2z)(1-6z)}. \\
&= z\left(\frac{1}{(1+2z)(1-6(-\frac{1}{2}))} + \frac{1}{(1+2(\frac{1}{6}))(1-6z)}\right) \\
&\quad +z^2\left(\frac{1}{(1-z)(1+2)(1-6)} + \frac{1}{(1-(-\frac{1}{2}))(1+2z)(1-6(-\frac{1}{2}))} + \frac{1}{(1-\frac{1}{6})(1+2(\frac{1}{6}))(1-6z)}\right) \\
&= \frac{\frac{1}{4}z}{1+2z} + \frac{\frac{3}{4}z}{1-6z} + \frac{-\frac{1}{15}z^2}{1-z} + \frac{\frac{1}{6}z^2}{1+2z} + \frac{\frac{9}{10}z^2}{1-6z}.
\end{aligned}
$$

From this we can immediately read off the value of $T(n)$ for $n \geq 2$:

$$
\begin{aligned}
T(n) &= \frac{1}{4}(-2)^{n-1} + \frac{3}{4}6^{n-1} - \frac{1}{15} + \frac{1}{6}(-2)^{n-2} + \frac{9}{10}6^{n-2} \\
&= -\frac{1}{8}(-2)^n + \frac{1}{8}6^n - \frac{1}{15} + \frac{1}{24}(-2)^n + \frac{1}{40}6^n \\
&= \frac{3}{20}6^n - \frac{1}{12}(-2)^n - \frac{1}{15}.
\end{aligned}
$$

Let's check this against the solutions we get from the recurrence itself:

n	$T(n)$
0	0
1	1
2	$1 + 4 \cdot 1 + 12 \cdot 0 = 5$
3	$1 + 4 \cdot 5 + 12 \cdot 1 = 33$
4	$1 + 4 \cdot 33 + 12 \cdot 5 = 193$

We'll try $n = 3$, and get $T(3) = (3/20) \cdot 216 + 8/12 - 1/15 = (3 \cdot 3 \cdot 216 + 40 - 4)/60 = (1944 + 40 - 4)/60 = 1980/60 = 33$.

To be extra safe, let's try $T(2) = (3/20) \cdot 36 - 4/12 - 1/15 = (3 \cdot 3 \cdot 36 - 20 - 4)/60 = (324 - 20 - 4)/60 = 300/60 = 5$. This looks good too.

The moral of this exercise? Generating functions can solve ugly-looking recurrences exactly, but you have to be very very careful in doing the math.

6.2 Partial fraction expansion with repeated roots

Let $a_n = 2a_{n-1} + n$, with some constant a_0. We'd like to find a closed-form formula for a_n.

As a test, let's figure out the first few terms of the sequence:

$$\begin{aligned}
a_0 &= a_0 \\
a_1 &= 2a_0 + 1 \\
a_2 &= 4a_0{+}2{+}2{=}\ 4a_0 + 4 \\
a_3 &= 8a_0{+}8{+}3{=}\ 8a_0 + 11 \\
a_4 &= 16a_0 + 22{+}4{=}\ 16a_0 + 26
\end{aligned}$$

The a_0 terms look nice (they're $2^n a_0$), but the 0, 1, 4, 11, 26 sequence doesn't look like anything familiar. So we'll find the formula the hard way.

First we convert the recurrence into an equation over generating functions and solve for the generating function F:

$$\begin{aligned}
\sum a_n z^n &= 2\sum a_{n-1} z^n + \sum n z^n + a_0 \\
F &= 2zF + \frac{z}{(1-z)^2} + a_0 \\
(1-2z)F &= \frac{z}{(1-z)^2} + a_0 \\
F &= \frac{z}{(1-z)^2(1-2z)} + \frac{a_0}{1-2z}.
\end{aligned}$$

Observe that the right-hand term gives us exactly the $2^n a_0$ terms we expected, since $1/(1-2z)$ generates the sequence 2^n. But what about the left-hand term? Here we need to apply a partial-fraction expansion, which is simplified because we already know how to factor the denominator but is complicated because there is a repeated root.

We can now proceed in one of two ways: we can solve directly for the partial fraction expansion, or we can use an extended version of Heaviside's cover-up method that handles repeated roots using differentiation. We'll start with the direct method.

6.2.1 Solving for the PFE directly

Write

$$\frac{1}{(1-z)^2(1-2z)} = \frac{A}{(1-z)^2} + \frac{B}{1-2z}$$

.

We expect B to be a constant and A to be of the form $A_1 z + A_0$.

To find B, use the technique of multiplying by $1-2z$ and setting $z = 1/2$:

$$\frac{1}{(1-\frac{1}{2})^2} = \frac{A \cdot 0}{(1-z)^2} + B.$$

So $B = 1/(1-1/2)^2 = 1/(1/4) = 4$.

We can't do this for A, but we can solve for it after substituting in $B = 4$:

$$\begin{aligned}
\frac{1}{(1-z)^2(1-2z)} &= \frac{A}{(1-z)^2} + \frac{4}{1-2z} \\
1 &= A(1-2z) + 4(1-z)^2 \\
A &= \frac{1-4(1-z)^2}{1-2z} \\
&= \frac{1-4+8z-4z^2}{1-2z} \\
&= \frac{-3+8z-4z^2}{1-2z} \\
&= \frac{-(1-2z)(3-2z)}{1-2z} \\
&= 2z-3.
\end{aligned}$$

So we have the expansion

$$\frac{1}{(1-z)^2(1-2z)} = \frac{2z-3}{(1-z)^2} + \frac{4}{1-2z},$$

from which we get

$$\begin{aligned}
F &= \frac{z}{(1-z)^2(1-2z)} + \frac{a_0}{1-2z} \\
&= \frac{2z^2-3z}{(1-z)^2} + \frac{4z}{1-2z} + \frac{a_0}{1-2z}.
\end{aligned}$$

If we remember that $1/(1-z)^2$ generates the sequence $x_n = n+1$ and $1/(1-2z)$ generates $x_n = 2^n$, then we can quickly read off the solution (for large n):

$$a_n = 2(n-1) - 3n + 4 \cdot 2^{n-1} + a_0 \cdot 2^n = 2^n a_0 + 2^{n+1} - 2 - n$$

which we can check by plugging in particular values of n and comparing it to the values we got by iterating the recurrence before.

The reason for the "large n" caveat is that $z^2/(1-z)^2$ doesn't generate precisely the sequence $x_n = n-1$, since it takes on the values $0, 0, 1, 2, 3, 4, \ldots$ instead of $-1, 0, 1, 2, 3, 4, \ldots$. Similarly, the power series for $z/(1-2z)$ does not have the coefficient $2^{n-1} = 1/2$ when $n = 0$. Miraculously, in this particular example the formula works for $n = 0$, even though it shouldn't: $2(n-1)$ is -2 instead of 0, but $4 \cdot 2^{n-1}$ is 2 instead of 0, and the two errors cancel each other out.

6.2.2 Solving for the PFE using the extended cover-up method

It is also possible to extend the cover-up method to handle repeated roots. Here we choose a slightly different form of the partial fraction expansion:

$$\frac{1}{(1-z)^2(1-2z)} = \frac{A}{(1-z)^2} + \frac{B}{1-z} + \frac{C}{1-2z}.$$

Here A, B, and C are all constants. We can get A and C by the cover-up method, where for A we multiply both sides by $(1-z)^2$ before setting $z=1$; this gives $A = 1/(1-2) = -1$ and $C = 1/(1-\frac{1}{2})^2 = 4$. For B, if we multiply both sides by $(1-z)$ we are left with $A/(1-z)$ on the right-hand side and a $(1-z)$ in the denominator on the left-hand side. Clearly setting $z=1$ in this case will not help us.

The solution is to first multiply by $(1-z)^2$ as before but then take a derivative:

$$\begin{aligned}
\frac{1}{(1-z)^2(1-2z)} &= \frac{A}{(1-z)^2} + \frac{B}{1-z} + \frac{C}{1-2z} \\
\frac{1}{1-2z} &= A + B(1-z) + \frac{C(1-z)^2}{1-2z} \\
\frac{d}{dz}\frac{1}{1-2z} &= \frac{d}{dz}\left(A + B(1-z) + \frac{C(1-z)^2}{1-2z}\right) \\
\frac{2}{(1-2z)^2} &= -B + \frac{-2C(1-z)}{1-2z} + \frac{2C(1-z)^2}{(1-2z)^2}
\end{aligned}$$

Now if we set $z=1$, every term on the right-hand side except $-B$ becomes 0, and we get $-B = 2/(1-2)^2$ or $B=-2$.

Plugging A, B, and C into our original formula then gives

$$\frac{1}{(1-z)^2(1-2z)} = \frac{-1}{(1-z)^2} + \frac{-2}{1-z} + \frac{4}{1-2z},$$

and thus

$$F = \frac{z}{(1-z)^2(1-2z)} + \frac{a_0}{1-2z} = z\left(\frac{-1}{(1-z)^2} + \frac{-2}{1-z} + \frac{4}{1-2z}\right) + \frac{a_0}{1-2z}.$$

From this we can read off (for large n):

$$a_n = 4 \cdot 2^{n-1} - n - 2 + a_0 \cdot 2^n = 2^{n+1} + 2^n a_0 - n - 2.$$

We believe this because it looks like the solution we already got.

7 Asymptotic estimates

We can simplify our life considerably if we only want an asymptotic estimate of a_n (see AsymptoticNotation). The basic idea is that if a_n is non-negative for sufficiently large n and $\sum a_n z^n$ converges for some fixed value z, then a_n must be $o(z^{-n})$ in the limit. (Proof: otherwise, $a_n z^n$ is at least a constant for infinitely many n, giving a divergent sum.) So we can use the **radius of convergence** of a generating function $F(z)$, defined as the largest value r such that $F(z)$ is defined for all (complex) z with $|z| < r$, to get a quick estimate of the growth rate of F's coefficients: whatever they do, we have $a_n = O(r^{-n})$.

For generating functions that are **rational functions** (ratios of polynomials), we can use the partial fraction expansion to do even better. First observe that for $F(z) = \sum f_i z^n = 1/(1-az)^k$, we have $f_n = \binom{k+n-1}{n} a^n = \frac{(n+k-1)(n+k-2)\dots(k-1)}{(k-1)!} a^n = \Theta(a^n n^{k-1})$. Second, observe that the numerator is irrelevant: if $1/(1-az)^k = \Theta(a^n n^{k-1})$ then $bz^m/(1-az)^{k-1} = b\Theta(a^{n-m}(n-m)^{k-1}) = ba^{-m}(1-m/n)^{k-1}\Theta(a^n n^{k-1}) = \Theta(a^n n^{k-1})$, because everything outside the Θ disappears into the constant for sufficiently large n. Finally, observe that in a partial fraction expansion, the term $1/(1-az)^k$ with the largest coefficient a (if there is one) wins in the resulting asymptotic sum: $\Theta(a^n) + \Theta(b^n) = \Theta(a^n)$ if $|a| > |b|$. So we have:

Theorem 1. *Let $F(z) = \sum f_n z^n = P(z)/Q(z)$ where P and Q are polynomials in z. If Q has a root r with multiplicity k, and all other roots s of Q satisfy $|r| < |s|$, then $f_n = \Theta((1/r)^n n^{k-1})$.*

The requirement that r is a unique minimal root of Q is necessary; for example, $F(z) = 2/(1-z^2) = 1/(1-z) + 1/(1+z)$ generates the sequence $0, 2, 0, 2, \dots$, which is *not* $\Theta(1)$ because of all the zeros; here the problem is that $1-z^2$ has two roots with the same absolute value, so for some values of n it is possible for them to cancel each other out.

A root in the denominator of a rational function F is called a **pole**. So another way to state the theorem is that the asymptotic value of the coefficients of a rational generating function is determined by the smallest pole.

More examples:

$F(z)$	Smallest pole	Asymptotic value
$1/(1-z)$	1	$\Theta(1)$
$1/(1-z)^2$	$1, multiplicity 2$	$\Theta(n)$
$1/(1-z-z^2)$	$(\sqrt{5}-1)/2 = 2/(1+\sqrt{5})$	$\Theta(((1+\sqrt{5})/2)^n)$
$1/((1-z)(1-2z)(1-3z))$	$1/3$	$\Theta(3^n)$
$(z+z^2(1-z))/(1-4z-12z^2)$	$1/6$	$\Theta(6^n)$
$1/((1-z)^2(1-2z))$	$1/2$	$\Theta(2^n)$

In each case it may be instructive to compare the asymptotic values to the exact values obtained earlier on this page.

8 Recovering the sum of all coefficients

Given a generating function for a convergent series $\sum_i a_i z^i$, we can compute the sum of all the a_i by setting z to 1. Unfortunately, for many common generating functions setting $z = 1$ yields $0/0$ (if it yields something else divided by zero then the series diverges). In this case we can recover the correct sum by taking the limit as z goes to 1 using *L'Hôpital's rule*, which says that $\lim_{x \to c} f(x)/g(x) = \lim_{x \to c} f'(x)/g'(x)$ when the latter limit exists and either $f(c) = g(c) = 0$ or $f(c) = g(c) = \infty$.[2]

[2] The justification for doing this is that we know that a finite sequence really has a finite sum, so the "singularity" appearing at $z = 1$ in e.g. $\frac{1-z^{n+1}}{1-z}$ is an artifact of the generating-

8.1 Example

Let's derive the formula for $1 + 2 + \ldots + n$. We'll start with the generating function for the series $\sum_{i=0}^{n} z^i$, which is $(1 - z^n + 1)/(1 - z)$. Applying the $z\frac{d}{dz}$ method gives us

$$\begin{aligned}\sum_{i=0}^{n} iz^i &= z\frac{d}{dz}\frac{1 - z^{n+1}}{1 - z} \\ &= z\left(\frac{1}{(1-z)^2} - \frac{(n+1)z^n}{1-z} - \frac{z^{n+1}}{(1-z)^2}\right) \\ &= \frac{z - (n+1)z^{n+1} + nz^{n+2}}{(1-z)^2}.\end{aligned}$$

Plugging $z = 1$ into this expression gives $(1-(n+1)+n)/(1-1) = 0/0$, which does not make us happy. So we go to the hospital—twice, since one application of L'Hôpital's rule doesn't get rid of our $0/0$ problem:

$$\begin{aligned}\lim_{z\to 1}\frac{z - (n+1)z^{n+1} + nz^{n+2}}{(1-z)^2} &= \lim_{z\to 1}\frac{1 - (n+1)^2 z^n + n(n+2)z^{n+1}}{-2(1-z)} \\ &= \lim_{z\to 1}\frac{-n(n+1)^2 z^{n-1} + n(n+1)(n+2)z^n}{2} \\ &= \frac{-n(n+1)^2 + n(n+1)(n+2)}{2} \\ &= \frac{-n^3 - 2n^2 - n + n^3 + 3n^2 + 2n}{2} \\ &= \frac{n^2 + n}{2} = \frac{n(n+1)}{2},\end{aligned}$$

which is our usual formula. Gauss's childhood proof is a lot quicker, but the generating-function proof is something that we could in principle automate most of the work using a computer algebra system, and it doesn't require much creativity or intelligence. So it might be the weapon of choice for nastier problems where no clever proof comes to mind.

More examples of this technique can be found on the BinomialCoefficients page, where the binomial theorem applied to $(1 + x)^n$ (which is really just a generating function for $\sum \binom{n}{i} z^i$) is used to add up various sums of binomial coefficients.

9 A recursive generating function

Let's suppose we want to count binary trees with n internal nodes. We can obtain such a tree either by (a) choosing an empty tree (g.f.: $z^0 = 1$); or (b) choosing a root with weight 1 (g.f. $1 \cdot z^1 = z$), since we can choose it in exactly

function representation rather than the original series—it's a "removable singularity" that can be replaced by the limit of $f(x)/g(x)$ as $x \to c$.

one way), and two subtrees (g.f. $= F^2$ where F is the g.f. for trees). This gives us a recursive definition

$$F = 1 + zF^2.$$

Solving for F using the quadratic formula gives

$$F = \frac{1 \pm \sqrt{1-4z}}{2z}.$$

That $2z$ in the denominator may cause us trouble later, but let's worry about that when the time comes. First we need to figure out how to extract coefficients from the square root term.

The binomial theorem says

$$\sqrt{1-4z} = (1-4z)^{1/2} = \sum_{n=0}^{\infty} \binom{1/2}{n} (-4z)^n.$$

For $n \geq 1$, we can expand out the $\binom{1/2}{n}$ terms as

$$\begin{aligned}
\binom{1/2}{n} &= \frac{(1/2)_{(n)}}{n!} \\
&= \frac{1}{n!} \cdot \prod_{k=0}^{n-1} (1/2 - k) \\
&= \frac{1}{n!} \cdot \prod_{k=0}^{n-1} \frac{1-2k}{2} \\
&= \frac{(-1)^n}{2^n n!} \cdot \prod_{k=0}^{n-1} (2k-1) \\
&= \frac{(-1)^n}{2^n n!} \cdot \frac{\prod_{k=1}^{2n-2} k}{\prod_{k=1}^{n-1} 2k} \\
&= \frac{(-1)^n}{2^n n!} \cdot \frac{(2n-2)!}{2^{n-1}(n-1)!} \\
&= \frac{(-1)^n}{2^{2n-1}} \cdot \frac{(2n-2)!}{n!(n-1)!} \\
&= \frac{(-1)^n}{2^{2n-1}(2n-1)} \cdot \frac{(2n-1)!}{n!(n-1)!} \\
&= \frac{(-1)^n}{2^{2n-1}(2n-1)} \cdot \binom{2n-1}{n}.
\end{aligned}$$

For $n = 0$, the switch from the big product of odd terms to $(2n-2)!$ divided by the even terms doesn't work, because $(2n-2)!$ is undefined. So here we just use the special case $\binom{1/2}{0} = 1$.

Now plug this nasty expression back into F to get

$$\begin{aligned}
F &= \frac{1 \pm \sqrt{1-4z}}{2z} \\
&= \frac{1}{2z} \pm \frac{1}{2z} \sum_{n=0}^{\infty} \binom{1/2}{n} (-4z)^n \\
&= \frac{1}{2z} \pm \left(\frac{1}{2z} + \frac{1}{2z} \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{2^{2n-1}(2n-1)} \binom{2n-1}{n} (-4z)^n \right) \\
&= \frac{1}{2z} \pm \left(\frac{1}{2z} + \frac{1}{2z} \sum_{n=1}^{\infty} \frac{(-1)^{2n-1} 2^{2n}}{2^{2n-1}(2n-1)} \binom{2n-1}{n} z^n \right) \\
&= \frac{1}{2z} \pm \left(\frac{1}{2z} + \frac{1}{2z} \sum_{n=1}^{\infty} \frac{-2}{(2n-1)} \binom{2n-1}{n} z^n \right) \\
&= \frac{1}{2z} \pm \left(\frac{1}{2z} - \sum_{n=1}^{\infty} \frac{1}{(2n-1)} \binom{2n-1}{n} z^{n-1} \right) \\
&= \frac{1}{2z} \pm \left(\frac{1}{2z} - \sum_{n=0}^{\infty} \frac{1}{(2n+1)} \binom{2n+1}{n+1} z^n \right) \\
&= \sum_{n=0}^{\infty} \frac{1}{(2n+1)} \binom{2n+1}{n+1} z^n \\
&= \sum_{n=0}^{\infty} \frac{1}{n+1} \binom{2n}{n} z^n.
\end{aligned}$$

Here we choose minus for the plus-or-minus to get the right answer and then do a little bit of tidying up of the binomial coefficient.

We can check the first few values of $f(n)$:

n	$f(n)$
0	$\binom{0}{0} = 1$
1	$(1/2)\binom{2}{1} = 1$
2	$(1/3)\binom{4}{2} = 6/3 = 2$
3	$(1/4)\binom{6}{3} = 20/4 = 5$

and these are consistent with what we get if we draw all the small binary trees by hand.

The numbers $\frac{1}{n+1}\binom{2n}{n}$ show up in a lot of places in combinatorics, and are known as the Catalan numbers.

10 Summary of operations on generating functions

The following table describes all the nasty things we can do to a generating function. Throughout, we assume $F = \sum f_k z^k$, $G = \sum g_k z^k$, etc.

Operation	**Effect on generating functions**	**Effect on coefficients**	**Combinatorial**
Coefficient extraction	$f_k = (1/k!) d^k/dz^k F(z)\|_{z=0}$		Find the number
Sum of all coefficients	$F(1)$	Computes $\sum f_k$	Count the total
Shift right	$G = zF$	$g_k = f_{k-1}$	Add 1 to the wei
Shift left	$G = z^{-1}(F - F(0))$	$g_k = f_{k+1}$	Subtract 1 from
Pointing	$G = z\frac{d}{dz}F$	$g_k = kf_k$	A G-thing is an
Sum	$H = F + G$	$h_k = f_k + g_k$	Every H-thing is
Product	$H = FG$	$h_k = \sum_i f_i g_{k-i}$	Every H-thing c
Composition	$H = F \circ G$	$H = \sum f_k G^k$	To make an H-th
Repetition	$G = 1/(1 - F)$	$G = \sum F^k$	A G-thing is a se

11 Variants

The **exponential generating function** or **egf** for a sequence $a_0, \ldots$ is given by $F(z) = \sum a_n z^n/n!$. For example, the egf for the sequence $1, 1, 1, \ldots$ is $e^z = \sum z^n/n!$. Exponential generating functions admit a slightly different set of operations from ordinary generating functions: differentiation gives left shift (since the factorials compensate for the exponents coming down), multiplying by z gives $b_n = na_{n+1}$, etc. The main application is that the product $F(z)G(z)$ of two egf's gives the sequence whose n-th term is $\sum (nchoosek) a_k b_{n-k}$; so for problems where we want that binomial coefficient in the convolution (e.g. when we are building weight n objects not only by choosing a weight-k object plus a weight-$(n-k)$ object but also by arbitrarily rearranging their unit-weight pieces) we want to use an egf rather than an ogf. We won't use these in CS202, but it's worth knowing they exist.

A **probability generating function** or **pgf** is essentially an ordinary generating function where each coefficient a_n is the probability that some random variable equals n. See RandomVariables for more details.

12 Further reading

RosenBook discusses some basic facts about generating functions in §7.4. ConcreteMathematics gives a more thorough introduction. Herbert Wilf's book *generatingfunctionology*, which can be downloaded from the web, will tell you more about the subject than you probably want to know.

See this page for very detailed notes on partial fraction expansion.

GENERATING FUNCTIONS AND RECURRENCE RELATIONS

Recurrence Relations

Suppose $a_0, a_1, a_2, \ldots, a_n, \ldots,$is an infinite sequence.
A recurrence recurrence relation is a set of equations

$$a_n = f_n(a_{n-1}, a_{n-2}, \ldots, a_{n-k}). \qquad (1)$$

The whole sequence is determined by (6) and the values of $a_0, a_1, \ldots, a_{k-1}$.

Linear Recurrence

Fibonacci Sequence

$$a_n = a_{n-1} + a_{n-2} \qquad n \geq 2.$$

$a_0 = a_1 = 1$.

$b_n = |B_n| = |\{x \in \{a, b, c\}^n : aa \text{ does not occur in } x\}|.$

$b_1 = 3 : \; a \; b \; c$

$b_2 = 8 : \; ab \; ac \; ba \; bb \; bc \; ca \; cb \; cc$

$b_n = 2b_{n-1} + 2b_{n-2} \qquad n \geq 2.$

$$b_n = 2b_{n-1} + 2b_{n-2} \qquad n \geq 2.$$

Let

$$B_n = B_n^{(b)} \cup B_n^{(c)} \cup B_n^{(a)}$$

where $B_n^{(\alpha)} = \{x \in B_n : \; x_1 = \alpha\}$ for $\alpha = a, b, c$.

Now $|B_n^{(b)}| = |B_n^{(c)}| = |B_{n-1}|$. The map $f : B_n^{(b)} \to B_{n-1}$,

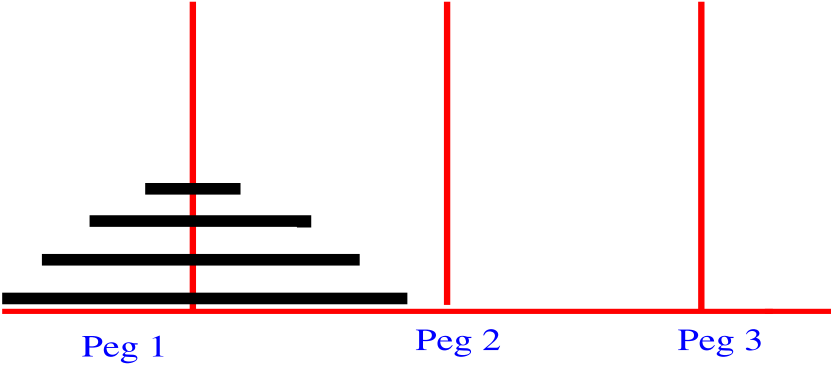
$$f(bx_2x_3 \ldots x_n) = x_2x_3 \ldots x_n \text{ is a bijection.}$$

$B_n^{(a)} = \{x \in B_n : \; x_1 = a$ and $x_2 = b$ or $c\}$. The map $g : B_n^{(a)} \to B_{n-1}^{(b)} \cup B_{n-1}^{(c)}$,

$$g(ax_2x_3 \ldots x_n) = x_2x_3 \ldots x_n \text{ is a bijection.}$$

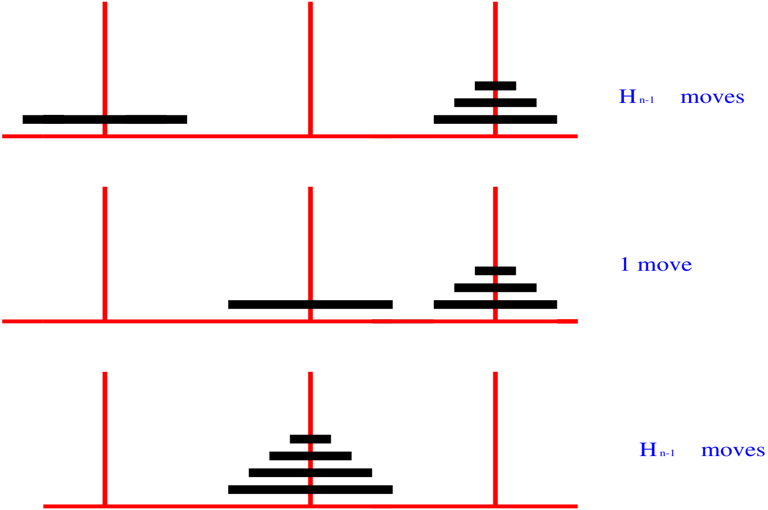
Hence, $|B_n^{(a)}| = 2|B_{n-2}|$.

Towers of Hanoi



H_n is the minimum number of moves needed to shift n rings from Peg 1 to Peg 2. One is not allowed to place a larger ring on top of a smaller ring.

xxx



A has n dollars. Everyday A buys one of a Bun (1 dollar), an Ice-Cream (2 dollars) or a Pastry (2 dollars). How many ways are there (sequences) for A to spend his money?
Ex. BBPIIPBI represents "Day 1, buy Bun. Day 2, buy Bun etc.".

$$\begin{aligned} u_n &= \text{number of ways} \\ &= u_{n,B} + u_{n,I} + u_{n,P} \end{aligned}$$

where $u_{n,B}$ is the number of ways where A buys a Bun on day 1 etc.

$u_{n,B} = u_{n-1}, \; u_{n,I} = u_{n,P} = u_{n-2}$.

So

$$u_n = u_{n-1} + 2u_{n-2},$$

and

$$u_0 = u_1 = 1.$$

If $a_0, a_1, \ldots, a_n$ is a sequence of real numbers then its **(ordinary) generating function** $a(x)$ is given by

$$a(x) = a_0 + a_1 x + a_2 x^2 + \cdots a_n x^n + \cdots$$

and we write

$$a_n = [x^n]a(x).$$

For more on this subject see Generatingfunctionology by the late Herbert S. Wilf. The book is available from https://www.math.upenn.edu// wilf/DownldGF.html

$a_n = 1$

$$a(x) = \frac{1}{1-x} = 1 + x + x^2 + \cdots + x^n + \cdots$$

$a_n = n + 1$.

$$a(x) = \frac{1}{(1-x)^2} = 1 + 2x + 3x^2 + \cdots + (n+1)x^n + \cdots$$

$a_n = n$.

$$a(x) = \frac{x}{(1-x)^2} = x + 2x^2 + 3x^3 + \cdots + nx^n + \cdots$$

Generalised binomial theorem:

$a_n = \binom{\alpha}{n}$

$$a(x) = (1+x)^\alpha = \sum_{n=0}^{\infty} \binom{\alpha}{n} x^n.$$

where

$$\binom{\alpha}{n} = \frac{\alpha(\alpha-1)(\alpha-2)\cdots(\alpha-n+1)}{n!}.$$

$a_n = \binom{m+n-1}{n}$

$$a(x) = \frac{1}{(1-x)^m} = \sum_{n=0}^{\infty} \binom{-m}{n} (-x)^n = \sum_{n=0}^{\infty} \binom{m+n-1}{n} x^n.$$

General view.

Given a recurrence relation for the sequence (a_n), we

(a) Deduce from it, an equation satisfied by the generating function $a(x) = \sum_n a_n x^n$.

(b) Solve this equation to get an explicit expression for the generating function.

(c) Extract the coefficient a_n of x^n from $a(x)$, by expanding $a(x)$ as a power series.

Solution of linear recurrences

$$a_n - 6a_{n-1} + 9a_{n-2} = 0 \qquad n \geq 2.$$

$a_0 = 1, a_1 = 9$.

$$\sum_{n=2}^{\infty}(a_n - 6a_{n-1} + 9a_{n-2})x^n = 0. \qquad (2)$$

$$
\begin{aligned}
\sum_{n=2}^{\infty} a_n x^n &= a(x) - a_0 - a_1 x \\
&= a(x) - 1 - 9x. \\
\sum_{n=2}^{\infty} 6a_{n-1} x^n &= 6x \sum_{n=2}^{\infty} a_{n-1} x^{n-1} \\
&= 6x(a(x) - a_0) \\
&= 6x(a(x) - 1). \\
\sum_{n=2}^{\infty} 9a_{n-2} x^n &= 9x^2 \sum_{n=2}^{\infty} a_{n-2} x^{n-2} \\
&= 9x^2 a(x).
\end{aligned}
$$

$$a(x) - 1 - 9x - 6x(a(x) - 1) + 9x^2 a(x) = 0$$

or

$$a(x)(1 - 6x + 9x^2) - (1 + 3x) = 0.$$

$$\begin{aligned} a(x) &= \frac{1+3x}{1-6x+9x^2} = \frac{1+3x}{(1-3x)^2} \\ &= \sum_{n=0}^{\infty}(n+1)3^n x^n + 3x\sum_{n=0}^{\infty}(n+1)3^n x^n \\ &= \sum_{n=0}^{\infty}(n+1)3^n x^n + \sum_{n=0}^{\infty} n3^n x^n \\ &= \sum_{n=0}^{\infty}(2n+1)3^n x^n. \end{aligned}$$

$$a_n = (2n+1)3^n.$$

Fibonacci sequence:

$$\sum_{n=2}^{\infty}(a_n - a_{n-1} - a_{n-2})x^n = 0.$$

$$\sum_{n=2}^{\infty} a_n x^n - \sum_{n=2}^{\infty} a_{n-1} x^n - \sum_{n=2}^{\infty} a_{n-2} x^n = 0.$$

$$(a(x) - a_0 - a_1 x) - (x(a(x) - a_0)) - x^2 a(x) = 0.$$

$$a(x) = \frac{1}{1 - x - x^2}.$$

$$
\begin{aligned}
a(x) &= -\frac{1}{(\xi_1 - x)(\xi_2 - x)} \\
&= \frac{1}{\xi_1 - \xi_2}\left(\frac{1}{\xi_1 - x} - \frac{1}{\xi_2 - x}\right) \\
&= \frac{1}{\xi_1 - \xi_2}\left(\frac{\xi_1^{-1}}{1 - x/\xi_1} - \frac{\xi_2^{-1}}{1 - x/\xi_2}\right)
\end{aligned}
$$

where

$$\xi_1 = -\frac{\sqrt{5}+1}{2} \text{ and } \xi_2 = \frac{\sqrt{5}-1}{2}$$

are the 2 roots of

$$x^2 + x - 1 = 0.$$

Therefore,

$$\begin{aligned} a(x) &= \frac{\xi_1^{-1}}{\xi_1 - \xi_2} \sum_{n=0}^{\infty} \xi_1^{-n} x^n - \frac{\xi_2^{-1}}{\xi_1 - \xi_2} \sum_{n=0}^{\infty} \xi_2^{-n} x^n \\ &= \sum_{n=0}^{\infty} \frac{\xi_1^{-n-1} - \xi_2^{-n-1}}{\xi_1 - \xi_2} x^n \end{aligned}$$

and so

$$\begin{aligned} a_n &= \frac{\xi_1^{-n-1} - \xi_2^{-n-1}}{\xi_1 - \xi_2} \\ &= \frac{1}{\sqrt{5}} \left(\left(\frac{\sqrt{5}+1}{2} \right)^{n+1} - \left(\frac{1-\sqrt{5}}{2} \right)^{n+1} \right). \end{aligned}$$

Inhomogeneous problem

$$a_n - 3a_{n-1} = n^2 \qquad n \geq 1.$$

$a_0 = 1$.

$$\begin{aligned}
\sum_{n=1}^{\infty}(a_n - 3a_{n-1})x^n &= \sum_{n=1}^{\infty} n^2 x^n \\
\sum_{n=1}^{\infty} n^2 x^n &= \sum_{n=2}^{\infty} n(n-1)x^n + \sum_{n=1}^{\infty} nx^n \\
&= \frac{2x^2}{(1-x)^3} + \frac{x}{(1-x)^2} \\
&= \frac{x+x^2}{(1-x)^3} \\
\sum_{n=1}^{\infty}(a_n - 3a_{n-1})x^n &= a(x) - 1 - 3xa(x) \\
&= a(x)(1-3x) - 1.
\end{aligned}$$

$$\begin{aligned} a(x) &= \frac{x+x^2}{(1-x)^3(1-3x)} + \frac{1}{1-3x} \\ &= \frac{A}{1-x} + \frac{B}{(1-x)^2} + \frac{C}{(1-x)^3} + \frac{D+1}{1-3x} \end{aligned}$$

where

$$\begin{aligned} x+x^2 &\cong A(1-x)^2(1-3x) + B(1-x)(1-3x) \\ &\qquad + C(1-3x) + D(1-x)^3. \end{aligned}$$

Then

$$A = -1/2,\ B = 0,\ C = -1,\ D = 3/2.$$

So

$$a(x) = \frac{-1/2}{1-x} - \frac{1}{(1-x)^3} + \frac{5/2}{1-3x}$$
$$= -\frac{1}{2}\sum_{n=0}^{\infty} x^n - \sum_{n=0}^{\infty}\binom{n+2}{2}x^n + \frac{5}{2}\sum_{n=0}^{\infty}3^n x^n$$

So

$$a_n = -\frac{1}{2} - \binom{n+2}{2} + \frac{5}{2}3^n$$
$$= -\frac{3}{2} - \frac{3n}{2} - \frac{n^2}{2} + \frac{5}{2}3^n.$$

General case of linear recurrence

$$a_n + c_1 a_{n-1} + \cdots + c_k a_{n-k} = u_n, \qquad n \geq k.$$

$u_0, u_1, \ldots, u_{k-1}$ are given.

$$\sum (a_n + c_1 a_{n-1} + \cdots + c_k a_{n-k} - u_n)\, x^n = 0$$

It follows that for some polynomial $r(x)$,

$$a(x) = \frac{u(x) + r(x)}{q(x)}$$

where

$$q(x) = 1 + c_1 x + c_2 x^2 + \cdots + c_k x^k = \prod_{i=1}^{k} (1 - \alpha_i x)$$

and $\alpha_1, \alpha_2, \ldots, \alpha_k$ are the roots of $p(x) = 0$ where $p(x) = x^k q(1/x) = x^k + c_1 x^{k-1} + \cdots + c_0$.

Products of generating functions

$$a(x) = \sum_{n=0}^{\infty} a_n x^n,\ b(x)) = \sum_{n=0}^{\infty} b_n x^n.$$

$$\begin{aligned} a(x)b(x) &= (a_0 + a_1 x + a_2 x^2 + \cdots) \times \\ &\qquad (b_0 + b_1 x + b_2 x^2 + \cdots) \\ &= a_0 b_0 + (a_0 b_1 + a_1 b_0) x + \\ &\qquad (a_0 b_2 + a_1 b_1 + a_2 b_0) x^2 + \cdots \\ &= \sum_{n=0}^{\infty} c_n x^n \end{aligned}$$

where

$$c_n = \sum_{k=0}^{n} a_k b_{n-k}.$$

Derangements

$$n! = \sum_{k=0}^{n} \binom{n}{k} d_{n-k}.$$

Explanation: $\binom{n}{k} d_{n-k}$ is the number of permutations with exactly k cycles of length 1. Choose k elements ($\binom{n}{k}$ ways) for which $\pi(i) = i$ and then choose a derangement of the remaining $n - k$ elements.
So

$$\begin{aligned} 1 &= \sum_{k=0}^{n} \frac{1}{k!} \frac{d_{n-k}}{(n-k)!} \\ \sum_{n=0}^{\infty} x^n &= \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} \frac{1}{k!} \frac{d_{n-k}}{(n-k)!} \right) x^n. \end{aligned} \tag{3}$$

Let

$$d(x) = \sum_{m=0}^{\infty} \frac{d_m}{m!} x^m.$$

From (3) we have

$$\begin{aligned} \frac{1}{1-x} &= e^x d(x) \\ d(x) &= \frac{e^{-x}}{1-x} \\ &= \sum_{n=0}^{\infty} \sum_{k=0}^{n} \left(\frac{(-1)^k}{k!} \right) x^n. \end{aligned}$$

So

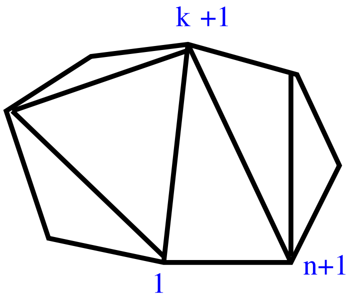
$$\frac{d_n}{n!} = \sum_{k=0}^{n} \frac{(-1)^k}{k!}.$$

Triangulation of n-gon

Let

$$a_n = \text{number of triangulations of } P_{n+1} = \sum_{k=0}^{n} a_k a_{n-k} \qquad n \geq 2 \tag{4}$$

$a_0 = 0$, $a_1 = a_2 = 1$.



Explanation of (4):

$a_k a_{n-k}$ counts the number of triangulations in which edge $1, n+1$ is contained in triangle $1, k+1, n+1$.

There are a_k ways of triangulating $1, 2, \ldots, k+1, 1$ and for each such there are a_{n-k} ways of triangulating $k+1, k+2, \ldots, n+1, k+1$.

$$x + \sum_{n=2}^{\infty} a_n x^n = x + \sum_{n=2}^{\infty} \left(\sum_{k=0}^{n} a_k a_{n-k} \right) x^n.$$

But,

$$x + \sum_{n=2}^{\infty} a_n x^n = a(x)$$

since $a_0 = 0, a_1 = 1$.

$$\begin{aligned} \sum_{n=2}^{\infty} \left(\sum_{k=0}^{n} a_k a_{n-k} \right) x^n &= \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} a_k a_{n-k} \right) x^n \\ &= a(x)^2. \end{aligned}$$

So

$$a(x) = x + a(x)^2$$

and hence

$$a(x) = \frac{1+\sqrt{1-4x}}{2} \text{ or } \frac{1-\sqrt{1-4x}}{2}.$$

But $a(0) = 0$ and so

$$\begin{aligned} a(x) &= \frac{1-\sqrt{1-4x}}{2} \\ &= \frac{1}{2} - \frac{1}{2}\left(1 + \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n2^{2n-1}} \binom{2n-2}{n-1} (-4x)^n\right) \\ &= \sum_{n=1}^{\infty} \frac{1}{n} \binom{2n-2}{n-1} x^n. \end{aligned}$$

So

$$a_n = \frac{1}{n}\binom{2n-2}{n-1}.$$

Exponential Generating Functions

Given a sequence $a_n, n \geq 0$, its exponential generating function (e.g.f.) $a_e(x)$ is given by

$$a_e(x) = \sum_{n=0}^{\infty} \frac{a_n}{n!} x^n$$

$$a_n = 1, n \geq 0 \text{ implies } a_e(x) = e^x.$$

$$a_n = n!, n \geq 0 \text{ implies } a_e(x) = \frac{1}{1-x}$$

Products of Exponential Generating Functions

Let $a_e(x), b_e(x)$ be the e.g.f.'s respectively for $(a_n), (b_n)$ respectively. Then

$$c_e(x) = a_e(x)b_e(x) = \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} \frac{a_k}{k!} \frac{b_{n-k}}{(n-k)!} \right) x^n$$

$$= \sum_{k=0}^{n} \frac{c_n}{n!} x^n$$

where

$$c_n = \binom{n}{k} a_k b_{n-k}.$$

Interpretation

Suppose that we have a collection of labelled objects and each object has a “size” k, where k is a non-negative integer. Each object is labelled by a set of size k.
Suppose that the number of labelled objects of size k is a_k.

Examples:
(a): Each object is a directed path with k vertices and its vertices are labelled by $1, 2, \ldots, k$ in some order. Thus $a_k = k!$.
(b): Each object is a directed cycle with k vertices and its vertices are labelled by $1, 2, \ldots, k$ in some order. Thus $a_k = (k-1)!$.

Now take example (a) and let $a_e(x) = \frac{1}{1-x}$ be the e.g.f. of this family. Now consider

$$c_e(x) = a_e(x)^2 = \sum_{n=0}^{\infty} (n+1)x^n \text{ with } c_n = (n+1) \times n!.$$

c_n is the number of ways of choosing an object of weight k and another object of weight $n-k$ and a partition of $[n]$ into two sets A_1, A_2 of size k and labelling the first object with A_1 and the second with A_2.

Here $(n+1) \times n!$ represents taking a permutation and choosing $0 \leq k \leq n$ and putting the first k labels onto the first path and the second $n-k$ labels onto the second path.

We will now use this machinery to count the number s_n of permutations that have an even number of cycles all of which have odd lengths:

Cycles of a permutation

Let $\pi : D \to D$ be a permutation of the finite set D. Consider the digraph $\Gamma_\pi = (D, A)$ where $A = \{(i, \pi(i)) : \ i \in D\}$. Γ_π is a collection of vertex disjoint cycles. Each $x \in D$ being on a unique cycle. Here a cycle can consist of a loop i.e. when $\pi(x) = x$.

Example: $D = [10]$.

i	1	2	3	4	5	6	7	8	9	10
$\pi(i)$	6	2	7	10	3	8	9	1	5	4

The cycles are $(1, 6, 8), (2), (3, 7, 9, 5), (4, 10)$.

In general consider the sequence $i, \pi(i), \pi^2(i), \ldots,$.

Since D is finite, there exists a first pair $k < \ell$ such that $\pi^k(i) = \pi^\ell(i)$. Now we must have $k = 0$, since otherwise putting $x = \pi^{k-1}(i) \neq y = \pi^{\ell-1}(i)$ we see that $\pi(x) = \pi(y)$, contradicting the fact that π is a permutation.

So i lies on the cycle $C = (i, \pi(i), \pi^2(i), \ldots, \pi^{k-1}(i), i)$.

If j is not a vertex of C then $\pi(j)$ is not on C and so we can repeat the argument to show that the rest of D is partitioned into cycles.

Now consider

$$a_e(x) = \sum_{m=0}^{\infty} \frac{(2m)!}{(2m+1)!} x^{2m+1}$$

Here

$$a_n = \begin{cases} 0 & n \text{ is even} \\ (n-1)! & n \text{ is odd} \end{cases}$$

Thus each object is an odd length cycle C, labelled by $[|C|]$.

Note that

$$\begin{aligned} a_e(x) &= \left(x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4} + \cdots\right) - \left(\frac{x^2}{2} + \frac{x^4}{4} + \cdots\right) \\ &= \log\left(\frac{1}{1-x}\right) - \frac{1}{2}\log\left(\frac{1}{1-x^2}\right) \\ &= \log\sqrt{\frac{1+x}{1-x}} \end{aligned}$$

Now consider $a_e(x)^\ell$. The coefficient of x^n in this series is $\frac{c_n}{n!}$ where c_n is the number of ways of choosing an ordered sequence of ℓ cycles of lengths $a_1, a_2, \ldots, a_\ell$ where $a_1 + a_2 + \cdots + a_\ell = n$. And then a partition of $[n]$ into $A_1, A_2, \ldots, A_\ell$ where $|A_i| = a_i$ for $i = 1, 2, \ldots, \ell$. And then labelling the ith cycle with A_i for $i = 1, 2, \ldots, \ell$.

We looked carefully at the case $\ell = 2$ and this needs a simple inductive step.

It follows that the coefficient of x^n in $\frac{a_e(x)^\ell}{\ell!}$ is $\frac{c_n}{n!}$ where c_n is the number of ways of choosing a set (unordered sequence) of ℓ cycles of lengths $a_1, a_2, \ldots, a_\ell$ . . .

What we therefore want is the coefficient of x^n in $1 + \frac{a_e(x)^2}{2!} + \frac{a(x)^4}{4!} + \cdots$.

Now

$$\sum_{k=0}^{\infty} \frac{a_e(x)^{2k}}{k!} = \frac{e^{a_e(x)} + e^{-a_e(x)}}{2} = \frac{1}{2}\left(\sqrt{\frac{1+x}{1-x}} + \sqrt{\frac{1-x}{1+x}}\right)$$

$$= \frac{1}{\sqrt{1-x^2}}$$

Thus

$$s_n = n![x^n]\frac{1}{\sqrt{1-x^2}} = \binom{n}{n/2}\frac{n!}{2^n}$$

Exponential Families

- P is a set referred to a set of pictures.
- A card C is a pair C, p, where $p \in P$ and S is a set of labels. The weight of C is $n = |S|$.
  If $S = [n]$ then C is a standard card.
- A hand H is a set of cards whose label sets form a partition of $[n]$ for some $n \geq 1$. The weight of H is n.
- $C' = (S', p)$ is a re-labelling of the card $C = (S, p)$ if $|S'| = |S|$.
- A deck $\mathcal{D}$ is a finite set of standard cards of common weight n, all of whose pictures are distinct.
- An exponential family $\mathcal{F}$ is a collection $\mathcal{D}_n, n \geq 1$, where the weight of $\mathcal{D}_n$ is n.

Given $\mathcal{F}$ let $h(n, k)$ denote the number of hands of weight n consisting of k cards, such that each card is a re-labelling of some card in some deck of $\mathcal{F}$.
(The same card can be used for re-labelling more than once.)
Next let the hand enumerator $\mathcal{H}(x, y)$ be defined by

$$\mathcal{H}(x, y) = \sum_{\substack{n \geq 1 \\ k \geq 0}} h(n, k) \frac{x^n}{n!} y^k, \qquad (h(n, 0) = \mathbf{1}_{n=0}).$$

Let $d_n = |\mathcal{D}_n|$ and $\mathcal{D}(x) = \sum_{n=1}^{\infty} \frac{d_n}{n!} x^n$.

Theorem

$$\mathcal{H}(x, y) = e^{y\mathcal{D}(x)}. \tag{5}$$

Example 1: Let $P = \{directed\ cycles\ of\ all\ lengths\}$.

A card is a directed cycle with labelled vertices.

A hand is a set of directed cycles of total length n whose vertex labels partition $[n]$ i.e. it corresponds to a permutation of $[n]$.

$d_n = (n-1)!$ and so

$$\mathcal{D}(x) = \sum_{n=1}^{\infty} \frac{x^n}{n} = \log\left(\frac{1}{1-x}\right)$$

and

$$\mathcal{H}(x,y) = \exp\left\{y \log\left(\frac{1}{1-x}\right)\right\} = \frac{1}{(1-x)^y}.$$

Let $\left[{n \atop k}\right]$ denote the number of permutations of $[n]$ with exactly k cycles. Then

$$\sum_{k=1}^{n} \left[{n \atop k}\right] y^k = \left[\frac{x^n}{n!}\right] \frac{1}{(1-x)^y}$$
$$= n! \binom{y+n-1}{n}$$
$$= y(y+1)\cdots(y+n-1).$$

The values $\left[{n \atop k}\right]$ are referred to as the Stirling numbers of the first kind.

Example 2: Let $P = \{[n], n \geq 1\}$.
A card is a non-empty set of positive integers.
A hand of k cards is a partition of $[n]$ into k non-empty subsets.
$d_n = 1$ for $n \geq 1$ and so

$$\mathcal{D}(x) = \sum_{n=1}^{\infty} \frac{x^n}{n!} = e^x - 1$$

and

$$\mathcal{H}(x, y) = e^{y(e^x - 1)}.$$

So, if $\left\{ {n \atop k} \right\}$ is the number of partitions of $[n]$ into k parts then

$$\left\{ {n \atop k} \right\} = \left[\frac{x^n}{n!} \right] \frac{(e^x - 1)^k}{k!}.$$

The values $\left\{ {n \atop k} \right\}$ are referred to as the Stirling numbers of the second kind.

Proof of (5)**:** Let $\mathcal{F}', \mathcal{F}''$ be two exponential families whose picture sets are disjoint. We merge them to form $\mathcal{F} = \mathcal{F}' \oplus \mathcal{F}''$ by taking all d'_n cards from the deck $\mathcal{D}'_n$ and adding them to the deck $\mathcal{D}''_n$ to make a deck of $d'_n + d''_n$ cards.

We claim that

$$\mathcal{H}(x,y) = \mathcal{H}'(x,y)\mathcal{H}''(x,y). \tag{6}$$

Indeed, a hand of $\mathcal{F}$ consists of k' cards of total weight n' together with $k'' = k - k'$ cards of total weight $n'' = n - n'$. The cards of $\mathcal{F}'$ will be labelled from an n'-subset S of $[n]$. Thus,

$$h(n,k) = \sum_{n',k'} \binom{n}{n'} h'(n',k')h''(n-n',k-k').$$

But,

$$\mathcal{H}'(x,y)\mathcal{H}''(x,y) = \left(\sum_{n',k'} h(n',k')\frac{x^{n'}}{n'!}y^{k'}\right)\left(\sum_{n'',k''} h(n'',k'')\frac{x^{n''}}{n''!}y^{k''}\right)$$
$$= \sum_{n,k}\left(\frac{n!}{n'(n-n')!}h(n',k')h(n'',k'')\right)\frac{x^n}{n!}y^k.$$

This implies (6).

Now fix positive weights r, d and consider an exponential family $\mathcal{F}_{r,d}$ that has d cards in deck $\mathcal{D}_r$ and no other non-empty decks. We claim that the hand enumerator of $\mathcal{F}_{r,d}$ is

$$\mathcal{H}_{r,d}(x,y) = \exp\left\{\frac{ydx^r}{r!}\right\}. \tag{7}$$

We prove this by induction on d.

Base Case $d = 1$**:** A hand consists of $k \geq 0$ copies of the unique standard card that exists. If $n = kr$ then there are

$$\binom{n!}{r!r!\cdots r!} = \frac{n!}{(r!)^k}$$

choices for the labels of the cards. Then

$$h(kr,k) = \frac{1}{k!}\frac{n!}{(r!)^k}$$

where we have divided by $k!$ because the cards in a hand are unordered. If r does not divide n then $h(n,k) = 0$.

Thus,

$$\mathcal{H}_{r,1}(x,y) = \sum_{k=0}^{\infty} \frac{1}{k!} \frac{n!}{(r!)^k} \frac{x^n}{n!} y^k$$
$$= \exp\left\{\frac{yx^r}{r!}\right\}$$

Inductive Step: $\mathcal{F}_{r,d} = \mathcal{F}_{r,1} \oplus \mathcal{F}_{r,d-1}$. So,

$$\mathcal{H}_{r,d}(x,y) = \mathcal{H}_{r,1}(x,y)\mathcal{H}_{r,d-1}(x,y)$$
$$= \exp\left\{\frac{yx^r}{r!}\right\} \exp\left\{\frac{y(d-1)x^r}{r!}\right\}$$
$$= \exp\left\{\frac{ydx^r}{r!}\right\},$$

completing the induction.

Now consider a general deck $\mathcal{D}$ as the union of disjoint decks $\mathcal{D}_r, r \geq 1$. then,

$$\mathcal{H}(x, y) = \prod_{r \geq 1} \mathcal{H}_r(x, y) = \prod_{r \geq 1} \exp\left\{\frac{yd x^r}{r!}\right\} = e^{y\mathcal{D}(x)}.$$

8.2 Solving Linear Recurrence Relations

Recall from Section 8.1 that solving a recurrence relation means to find explicit solutions for the recurrence relation.

Definition 1. *A linear homogeneous recurrence relation of degree k with constant coefficients is a recurrence relation of the form*

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k}, \qquad (*)$$

where $c_1, c_2, \ldots, c_k \in \mathbb{R}$ and $c_k \neq 0$.

Linear refers to the fact that $a_{n-1}, a_{n-2}, \ldots, a_{n-k}$ appear in separate terms and to the first power.

Homogeneous refers to the fact that the total degree of each term is the same (thus there is no constant term)

Constant Coefficients refers to the fact that $c_1, c_2, \ldots, c_k$ are fixed real numbers that do not depend on n.

Degree k refers to the fact that the expression for a_n contains the previous k terms $a_{n-1}, a_{n-2}, \ldots, a_{n-k}$.

A consequence of the second principle of mathematical induction is that a sequence satisfying the recurrence relation in the definition (*) is uniquely determined once we know the values of a_j in the k initial conditions

$$a_0 = C_0, a_1 = C_1, \ldots, a_{k-1} = C_{k-1}.$$

Example 1. *The recurrence relation $A_n = (1.04)A_{n-1}$ is a linear homogeneous recurrence relation of degree one. The recurrence relation $F_n = F_{n-1} + F_{n-2}$ is a linear homogeneous recurrence relation of degree two. The recurrence relation $a_n = a_{n-5}$ is a linear homogeneous recurrence relation of degree five.*

Example 2 (Non-examples)**.** *The recurrence relation $a_n = a_{n-1}a_{n-2}$ is not linear. The recurrence relation $m_n = 2m_{n-1} + 1$ is not homogeneous. The recurrence relation $B_n = nB_{n-1}$ does not have constant coefficients.*

Linear homogeneous recurrence relations are studied for two reasons. First, they often occur in modeling of problems. Second, they can be systematically solved. The basic approach for solving linear homogeneous recurrence relations is to look for solutions of the form $a_n = r^n$, where r is a constant.

Remark 1. *Note that $a_n = r^n$ is a solution of the recurrence relation (*) if and only if*

$$r^n = c_1 r^{n-1} + c_2 r^{n-2} + \cdots + c_k r^{n-k}.$$

Divide both sides of the above equation by r^{n-k} and subtract the right-hand side from the left to obtain

$$r^k - c_1 r^{k-1} - c_2 r^{k-2} - \cdots c_k = 0. \qquad (**)$$

Consequently, the sequence $\{a_n\}$ with $a_n = r^n$ is a solution of (*) if and only if r is a solution of (**).

Definition 2. *We call the equation*

$$r^k - c_1 r^{k-1} - c_2 r^{k-2} - \cdots c_k = 0. \qquad (**)$$

the characteristic equation *of the recurrence relation (*). The solutions of this equation are called the* characteristic roots *of the recurrence relation (*).*

As we will see, these characteristic roots can be used to give an explicit formula for all the solutions of the recurrence relation.

We first consider the case of degree two.

The Distinct-Roots Case

Consider a second-order linear homogeneous recurrence relation with constant coefficients:

$$a_k = Aa_{k-1} + Ba_{k-2} \quad \text{for all integers } k \geq 2, \tag{1}$$

where A and B are fixed real numbers. Relation (1) is satisfied when all the $a_i = 0$, but it has nonzero solutions as well. *Suppose* that for some number t with $t \neq 0$, the sequence

$$1, t, t^2, \ldots, t^n, \ldots$$

satisfies relation (1). This means that each term of the sequence equals A times the previous term plus B times the term before that. So for all integers $k \geq 2$,

$$t^k = At^{k-1} + Bt^{k-2}.$$

In particular, when $k = 2$, the equation becomes

$$t^2 = At + B,$$

or equivalently,

$$t^2 - At - B = 0. \tag{2}$$

This is a quadratic equation, and the values of t that make it true can be found either by factoring or by using the quadratic formula.

Now work backward. *Suppose* t is any number that satisfies equation (2). Does the sequence $1, t, t^2, t^3, \ldots, t^n, \ldots$ satisfy relation (1)? To answer this question, multiply equation (2) by t^{k-2} to obtain

$$t^{k-2} \cdot t^2 - t^{k-2} \cdot At - t^{k-2} \cdot B = 0.$$

This is equivalent to

$$t^k - At^{k-1} - Bt^{k-2} = 0$$

or

$$t^k = At^{k-1} + Bt^{k-2}.$$

Hence the answer is yes: $1, t, t^2, t^3, \ldots, t^n, \ldots$ satisfies relation (1).

This discussion proves the following lemma.

Lemma 1. *Let A and B be real numbers. A recurrence relation of the form*

$$a_k = Aa_{k-1} + Ba_{k-2} \quad \textit{for all integers } k \geq 2, \tag{1}$$

is satisfied by the sequence

$$1, t, t^2, t^3, \ldots, t^n, \ldots,$$

where t is a nonzero real number, if, and only if, t satisfies the equation

$$t^2 - At - B = 0. \tag{2}$$

Lemma 2. *If $r_0, r_1, r_2, \ldots$ and $s_0, s_1, s_2, \ldots$ are sequences that satisfy the same second-order linear homogeneous recurrence relation with constant coefficients, and if C and D are any numbers, then the sequence $a_0, a_1, a_2, \ldots$ defined by the formula*

$$a_n = Cr^n + Ds^n \quad \textit{for all integers } n \geq 0$$

also satisfies the same recurrence relation.

Given a second-order linear homogeneous recurrence relation with constant coefficients, if the characteristic equation has two distinct roots, then Lemmas 1 and 2 can be used to find an explicit formula for *any* sequence that satisfies a second-order linear homogeneous recurrence relation with constant coefficients for which the characteristic equation has distinct roots, provided that the first two terms of the sequence are known. This is made precise in the next theorem.

Theorem 1 (Distinct Roots Theorem)**.** *Suppose a sequence* $a_0, a_1, a_2, \dots$ *satisfies a recurrence relation*

$$a_k = Aa_{k-1} + Ba_{k-2} \quad \text{for all integers } k \geq 2, \tag{1}$$

for some real numbers A *and* B *with* $B \neq 0$*. If the characteristic equation*

$$t^2 - At - B = 0 \tag{2}$$

has two distinct roots r *and* s*, then* $a_0, a_1, a_2, \dots$ *is given by the explicit formula*

$$a_n = Cr^n + Ds^n,$$

where C *and* D *are the numbers whose values are determined by the values* a_0 *and* a_1*.*

Remark 2. *To say "*C *and* D *are determined by the values of* a_0 *and* a_1*" means that* C *and* D *are the solutions to the system of simultaneous equations*

$$a_0 = Cr^0 + Ds^0 \text{ and } a_1 = Cr^1 + Ds^1,$$

or, equivalently,

$$a_0 = C + D \text{ and } a_1 = Cr + Ds.$$

This system always has a solution when $r \neq s$*.*

Proof. Suppose that for some real numbers A and B, a sequence $a_0, a_1, a_2, \dots$satisfies the recurrence relation $a_k = Aa_{k-1} + Ba_{k-2}$, for all integers $k \geq 2$, and suppose the characteristic equation $t^2 - At - B = 0$ has two distinct roots r and s. We will show that

$$\text{for all integers } n \geq 0, \quad a_n = Cr^n + Ds^n,$$

where C and D are numbers such that

$$a_0 = Cr^0 + Ds^0 \text{ and } a_1 = Cr^1 + Ds^1.$$

Let $P(n)$ be the equation

$$a_n = Cr^n + Ds^n.$$

We use strong mathematical induction to prove that $P(n)$ is true for all integers $n \geq 0$. In the basis step, we prove that $P(0)$ and $P(1)$ are true. We do this because in the inductive step we need the equation to hold for $n = 0$ and $n = 1$ in order to prove that it holds for $n = 2$.

Show that $P(0)$ **and** $P(1)$ **are true:** The truth of $P(0)$ and $P(1)$ is automatic because C and D are exactly those numbers that make the following equations true:

$$a_0 = Cr^0 + Ds^0 \text{ and } a_1 = Cr^1 + Ds^1.$$

Show that for all integers $k \geq 1$**, if** $P(i)$ **is true for all integers** i **from** 0 **through** k**, then** $P(k+1)$ **is also true:** Suppose that $k \geq 1$ and for all integers i from 0 through k,

$$a_i = Cr^i + Ds^i.$$

We must show that $P(k+1)$:

$$a_{k+1} = Cr^{k+1} + Ds^{k+1}.$$

Now by the inductive hypothesis,

$$a_k = Cr^k + Ds^k \text{ and } a_{k-1} = Cr^{k-1} + Ds^{k-1},$$

so

$$\begin{aligned} a_{k+1} &= Aa_k + Ba_{k-1} \\ &= A(Cr^k + Ds^k) + B(Cr^{k-1} + Ds^{k-1}) \\ &= C(Ar^k + Br^{k-1}) + D(As^k + Bs^{k-1}) \\ &= Cr^{k+1} + Ds^{k+1}. \end{aligned}$$

This is what was to be shown. [The reason the last equality follows from Lemma 1 is that since r and s satisfy the characteristic equation (2), the sequences $r^0, r^1, r^2, \ldots$ and $s^0, s^1, s^2, \ldots$ satisfy the recurrence relation (1).] □

Example 3. *The Fibonacci sequence $F_0, F_1, F_2, \ldots$ satisfies the recurrence relation*

$$F_k = F_{k-1} + F_{k-2} \quad \textit{for all integersk} \geq 2$$

with initial conditions

$$F_0 = F_1 = 1.$$

Find an explicit formula for this sequence.

Solution. The Fibonacci sequence satisfies part of the hypothesis of the distinct-roots theorem since the Fibonacci relation is a second-order linear homogeneous recurrence relation with constant coefficients ($A = 1$ and $B = 1$). Is the second part of the hypothesis also satisfied? Does the characteristic equation

$$t^2 - t - 1 = 0$$

have distinct roots? By the quadratic formula, the roots are

$$t = \frac{1 \pm \sqrt{1 - 4(-1)}}{2} = \begin{cases} \frac{1+\sqrt{5}}{2} \\ \frac{1-\sqrt{5}}{2} \end{cases}$$

and so the answer is yes. It follows from the distinct-roots theorem that the Fibonacci sequence is given by the explicit formula

$$F_n = C\left(\frac{1+\sqrt{5}}{2}\right)^n + D\left(\frac{1-\sqrt{5}}{2}\right)^n \quad \text{for all integers } n \geq 0, \tag{3}$$

where C and D are the numbers whose values are determined by the fact that $F_0 = F_1 = 1$. To find C and D, write

$$F_0 = 1 = C\left(\frac{1+\sqrt{5}}{2}\right)^0 + D\left(\frac{1-\sqrt{5}}{2}\right)^0 = C \cdot 1 + D \cdot 1 = C + D$$

and

$$F_1 = 1 = C\left(\frac{1+\sqrt{5}}{2}\right)^1 + D\left(\frac{1-\sqrt{5}}{2}\right)^1 = C\left(\frac{1+\sqrt{5}}{2}\right) + D\left(\frac{1-\sqrt{5}}{2}\right).$$

Thus the problem is to find numbers C and D such that

$$C + D = 1$$

and

$$C\left(\frac{1+\sqrt{5}}{2}\right)+D\left(\frac{1-\sqrt{5}}{2}\right)=1.$$

This may look complicated, but in fact it is just a system of two equations in two unknowns. The solutions are

$$C=\frac{1+\sqrt{5}}{2\sqrt{5}} \text{ and } D=\frac{-(1-\sqrt{5})}{2\sqrt{5}}.$$

Substituting these values for C and D into formula (3) gives

$$F_n=\left(\frac{1+\sqrt{5}}{2\sqrt{5}}\right)\left(\frac{1+\sqrt{5}}{2}\right)^n+\left(\frac{-(1-\sqrt{5})}{2\sqrt{5}}\right)\left(\frac{1-\sqrt{5}}{2}\right)^n,$$

or, simplifying,

$$F_n=\frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^{n+1}-\frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^{n+1} \tag{4}$$

for all integers $n \geq 0$. Remarkably, even though the formula for F_n involves $\sqrt{5}$, all of the values of the Fibonacci sequence are integers. □

Theorem 1 does not work when characteristic equation has double root. In this case, ...

The Single-Root Case

Consider again the recurrence relation

$$a_k = Aa_{k-1} + Ba_{k-2} \quad \text{for all integers } k \geq 2, \tag{1}$$

where A and B are real numbers, but suppose now that the characteristic equation

$$t^2 - At - B = 0. \tag{2}$$

has a single real root r. By Lemma 1, one sequence that satisfies the recurrence relation is

$$1, r, r^2, r^3, \ldots, r^n, \ldots$$

But another sequence that also satisfies the relation is

$$0, r, 2r^2, 3r^3, \ldots, nr^n, \ldots$$

To see why this is so, observe that since r is the unique root of $t^2 - At - B = 0$, the left-hand side of the equation can be factored as $(t-r)^2$, and so

$$t^2 - At - B = (t-r)^2 = t^2 - 2rt + r^2. \tag{5}$$

Equating coefficients in equation (5) gives

$$A = 2r \text{ and } B = -r^2. \tag{6}$$

Let $s_0, s_1, s_2, \ldots$ be the sequence defined by the formula

$$s_n = nr^n \quad \text{for all integers } n \geq 0.$$

Then

$$\begin{aligned}
As_{k-1} + Bs_{k-2} &= A(k-1)r^{k-1} + B(k-2)r^{k-2} \\
&= 2r(k-1)r^{k-1} - r^2(k-2)r^{k-2} \\
&= 2(k-1)r^k - (k-2)r^k \\
&= (2k-2-k+2)r^k \\
&= kr^k \\
&= s_k.
\end{aligned}$$

Thus $s_0, s_1, s_2, \ldots$ satisfies the recurrence relation. This argument proves the following lemma.

Lemma 3. *Let A and B be real numbers and suppose the characteristic equation*

$$t^2 - At - B = 0. \tag{2}$$

has a single root r. Then the sequences $1, r, r^2, r^3, \ldots, r^n, \ldots$ and $0, r, 2r^2, 3r^3, \ldots, nr^n, \ldots$ both satisfy the recurrence relation

$$a_k = Aa_{k-1} + Ba_{k-2} \quad \text{for all integers } k \geq 2, \tag{1}$$

Lemmas 2 and 3 can be used to establish the single-root theorem, which shows how to find an explicit formula for any recursively defined sequence satisfying a second-order linear homogeneous recurrence relation with constant coefficients for which the characteristic equation has just one root. Taken together, the distinct-roots and single-root theorems cover all second-order linear homogeneous recurrence relations with constant coefficients. The proof of the single-root theorem is very similar to that of the distinct-roots theorem.

Theorem 2 (Single-Root Theorem)**.** *Suppose a sequence $a_0, a_1, a_2, \ldots$ satisfies a recurrence relation*

$$a_k = Aa_{k-1} + Ba_{k-2} \quad \text{for all integers } k \geq 2, \tag{1}$$

for some real numbers A and B with $B \neq 0$. If the characteristic equation

$$t^2 - At - B = 0 \tag{2}$$

has a single (real) root r, then $a_0, a_1, a_2, \ldots$ is given by the explicit formula

$$a_n = Cr^n + Dnr^n,$$

where C and D are the numbers whose values are determined by the values a_0 and any other known value of the sequence.

Example 4. *Suppose a sequence $b_0, b_1, b_2, \ldots$ satisfies the recurrence relation*

$$b_k = 4b_{k-1} - 4b_{k-2} \quad \text{for all integers } k \geq 2, \tag{7}$$

with initial conditions

$$b_0 = 1 \text{ and } b_1 = 3.$$

Find an explicit formula for $b_0, b_1, b_2, \ldots$.

Solution. This sequence satisfies part of the hypothesis of the single-root theorem because it satisfies a second-order linear homogeneous recurrence relation with constant coefficients ($A = 4$ and $B = -4$). The single-root condition is also met because the characteristic equation

$$t^2 - 4t + 4 = 0$$

has the unique root $r = 2$ [since $t^2 - 4t + 4 = (t-2)^2$].

It follows from the single-root theorem that $b_0, b_1, b_2, \ldots$ is given by the explicit formula

$$b_n = C \cdot 2^n + Dn2^n \quad \text{for all integers } n \geq 0, \tag{8}$$

where C and D are the real numbers whose values are determined by the fact that $b_0 = 1$ and $b_1 = 3$. To find C and D, write

$$b_0 = C \cdot 2^0 + D \cdot 0 \cdot 2^0 = C$$

and

$$b_1 = C \cdot 2^1 + D \cdot 1 \cdot 2^1 = 2C + 2D.$$

Hence the problem is to find numbers C and D such that

$$C = 1$$

and

$$2C + 2D = 3.$$

Substitute $C = 1$ into the second equation to obtain

$$2 + 2D = 3,$$

and so

$$D = \frac{1}{2}.$$

Now substitute $C = 1$ and $D = \frac{1}{2}$ into formula (8) to conclude that

$$b_n = 2^n + \frac{1}{2}n2^n = 2^n\left(1 + \frac{n}{2}\right) \quad \text{for all integers } n \geq 0. \qquad \square$$

Theorem 3. *Let $c_1, c_2, \ldots, c_k$ be real numbers. Suppose that the characteristic equation*

$$r^k - c_1 r^{k-1} - c_2 r^{k-2} - \cdots - c_k = 0$$

has k distinct roots $r_1, r_2, \ldots, r_k$. Then a sequence $\{a_n\}$ is a solution of the recurrence relation

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k}$$

if and only if

$$a_n = A_1 r_1^n + A_2 r_2^n + \cdots + A_k r_k^n$$

for $n = 0, 1, 2, \ldots$, where $A_1, A_2, \ldots$ are constants.

Example 5. *Find the solution to the recurrence relation*

$$a_n = 6a_{n-1} - 11a_{n-2} + 6a_{n-3}$$

with the initial conditions $a_0 = 2, a_1 = 5,$ and $a_2 = 15$.

Solution. The characteristic polynomial of this recurrence relation is

$$r^3 - 6r^2 + 11r - 6$$

By the rational root test, the possible roots are $\pm 1, \pm 2, \pm 3, \pm 6$. We find that $r = 1$ is a root. We find the other roots by dividing $r - 1$ into $r^3 - 6r^2 + 11r - 6$. The characteristic roots are $r_1 = 1, r_2 = 2,$ and $r_3 = 3$. Hence, the solutions to this recurrence relation are of the form

$$a_n = A \cdot 1^n + B \cdot 2^n + C \cdot 3^n.$$

To find the constants A, B, and C, use the initial conditions. This gives

$$\begin{aligned} a_0 = 2 &\qquad = A + B + C \\ a_1 = 5 &\qquad = A + 2B + 3C \\ a_2 = 15 &\qquad = A + 4B + 9C \end{aligned}$$

When these three simultaneous equations are solved for A, B, and C, we find that $A = 1, B = -1$, and $C = 2$. Hence, the unique solution to this recurrence relation and the given initial conditions is the sequence $\{a_n\}$ with

$$a_n = 1 - 2^n + 2 \cdot 3^n.$$

□

Theorem 4 gives an analogue of Theorem 3 where roots can have multiplicity.

Theorem 4. *Let $c_1, c_2, \ldots, c_k$ be real numbers. Suppose that the characteristic equation*

$$r^k - c_1 r^{k-1} - \cdots - c_k = 0$$

has t distinct roots $r_1, r_2, \ldots, r_t$ with multiplicities $m_1, m_2, \ldots, m_t$ respectively, so that $m_i \geq 1$ for $i = 1, 2, \ldots, t$ and $m_1 + m_2 + \cdots + m_t = k$. Then a sequence $\{a_n\}$ is a solution of the recurrence relation

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k}$$

if and only if

$$\begin{aligned} a_n = {} & (\alpha_{1,0} + \alpha_{1,1} n + \cdots + \alpha_{1,m_1-1} n^{m_1-1}) r_1^n \\ & + (\alpha_{2,0} + \alpha_{2,1} n + \cdots + \alpha_{2,m_2-1} n^{m_2-1}) r_2^n \\ & + \cdots \\ & + (\alpha_{t,0} + \alpha_{t,1} n + \cdots + \alpha_{t,m_t-1} n^{m_t-1}) r_t^n \end{aligned}$$

for $n = 0, 1, 2, \ldots$, where $\alpha_{i,j}$ are constants for $1 \leq i \leq t$ and $0 \leq j \leq m_i - 1$.

8.3 Divide-and-Conquer Algorithms and Recurrence Relations

Divide and Conquer Algorithm: A divide and conquer algorithm works by recursively breaking down a problem into two or more sub-problems of the same (or related) type, until these become simple enough to be solved directly. The solutions to the sub-problems are then combined to give a solution to the original problem. - Wikipedia

Theorem 1: Let f be an increasing function that satisfies the recurrence relation

$$f(n) = af(n/b) + c$$

whenever n is divisible by b, where $a \geq 1$, b is an integer greater than 1, and c is a positive real number. Then

$$f(n) = \begin{cases} O(n^{\log_b a}) & \text{if } a > 1 \\ O(\log n) & \text{if } a = 1 \end{cases}$$

Furthermore, when $n = b^k$ and $a \neq 1$, where k is a positive integer,

$$f(n) = C_1 n^{\log_b a} + C_2$$

where $C_1 = f(1) + c/(a-1)$ and $C_2 = -c/(a-1)$.

Master Theorem: Let f be an increasing function that satisfies the recurrence relation

$$f(n) = af(n/b) + cn^d$$

whenever $n = b^k$, where k is a positive integer, $a \geq 1$, b is an integer greater than 1, and c and d are real numbers with c positive and d nonnegative. Then

$$f(n) = \begin{cases} O(n^d) & \text{if } a < b^d \\ O(n^d \log n) & \text{if } a = b^d \\ O(n^{\log_b a}) & \text{if } a > b^d \end{cases}$$

8.3 pg. 535 # 9

Suppose that $f(n) = f(n/5) + 3n^2$ when n is a positive integer divisible by 5, and $f(1) = 4$. Find

a $f(5)$.

Just have to work up to the desired number.

$f(5) = f(5/5) + 3(5)^2$
$= f(1) + 3(25)$
$= 4 + 75$
$= 79$

b $f(125)$.

$f(25) = f(25/5) + 3(25)^2$
$= f(5) + 3(625)$

$= 79 + 1875$
$= 1954$

$f(125) = f(125/5) + 3(125)^2$
$= f(25) + 3(15625)$
$= 1954 + 46875$
$= 48829$

c $f(3125)$.

$f(625) = f(625/5) + 3(625)^2$
$= f(125) + 3(390625)$
$= 48829 + 1171875$
$= 1220704$

$f(3125) = f(3125/5) + 3(3125)^2$
$= f(625) + 3(9765625)$
$= 1220704 + 29296875$
$= 30517579$

8.3 pg. 535 # 11

Give a big-O estimate for the function $f(n) = f(n/2) + 1$ if f is an increasing function and $n = 2^k$.

Use Master Theorem with $a = 1, b = 2, c = 1, d = 0$. Since $a = b^d$, we know that $f(n)$ is $O(n^d \log n) = O(\log n)$.

8.3 pg. 535 # 13

Give a big-O estimate for the function $f(n) = 2f(n/3) + 4$ if f is an increasing function and $n = 3^k$.

Use Master Theorem with $a = 2, b = 3, c = 4, d = 0$. Since $a > b^d$, we know that $f(n)$ is $O(n^{\log_b a}) = O(n^{\log_3 2})$.

8.3 pg. 535 # 17

Suppose that the votes of n people for different candidates (where there can be more than two candidates) for a particular office are the elements of a sequence. A person wins the election if this person receives a majority of the votes.

a Devise a divide-and-conquer algorithm that determines whether a candidate received a majority and, if so, determine who this candidate is. [*Hint:* Assume that n is even and split the sequence of votes into two sequences, each with $n/2$ elements. Note that a candidate could not received a majority of votes without receiving a majority of votes in at least one of the two halves.]

We will use the hint to devise our algorithm. We first note that our base case is that a sequence with one element means that the one person on the list is the winner. For our recursive step, we will divide the list into two equal parts and count which name occurs the most in the two parts. We know that the winner requires a majority of votes and that would mean that the winner will need at least half+1 of a part to be his/her name. Keep applying this recursive step to each half and we will eventually have only at most two names in the list. Count the number of occurrences in the whole list of the two remaining names and this will decide the winner. This will only require at most $2n$ additional comparisons for a list of length n.

b Use the master theorem to give a big-O estimate for the number of comparisons needed by the algorithm you devised in part (a).

Our function is $f(n) = 2f(n/2) + 2n$. So $a = 2, b = 2, c = 2, d = 1$. By the Master theorem, $a = b^d$ so, the big-O is $O(n^d \log n) = O(n \log n)$.